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Diophantine Inequalities with an Infinity of Solutions

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DIOPHANTINE INEQUALITIES WITH AN INFINITY OF SOLUTIONS

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CONTENTS

Many results in the geometry of numbers assert, in effect, that inequalities of a certain type are soluble in integers, the constant on the right of the inequality being the best possible. Recent work of Mahler often enables one to prove that such an inequality has infinitely many solutions. In this paper we develop the theory of inequalities with infinitely many solutions, and investigate more deeply some of the questions which naturally arise.

1. Introduction

Let $F(x_1, \ldots, x_n)$ be a continuous real function of n real variables satisfying

$$F(tx_1, \ldots, tx_n) = t^h F(x_1, \ldots, x_n) \tag{1}$$

for all real t, where h is a positive integer. Let $x_1, \ldots, x_n$ be any real linear forms in $u_1, \ldots, u_n$ of determinant 1. There are many theorems in the geometry of numbers which assert, when they are expressed in arithmetical form, that for a particular function $F(x_1, \ldots, x_n)$ and a particular number λ the inequality

$$|F(x_1, \ldots, x_n)| \leqslant \lambda \tag{2}$$

is always *soluble*, that is, has always at least one solution in integral values of the variables $u_1, \ldots, u_n$ other than $0, \ldots, 0$. Sometimes the proof is such that, by a simple variation of certain parameters, one can deduce the existence of an infinity of solutions. Suppose, for instance, that the inequality in question is

$$|x_1 \ldots x_n| \leqslant \lambda. \tag{3}$$

It was proved by Minkowski, as one of the simple applications of his fundamental theorem in the geometry of numbers, that the inequality

$$|x_1| + \ldots + |x_n| \leqslant \sqrt[n]{(n!)}$$

is always soluble. It follows from the inequality of the arithmetic and geometric means that (3) is soluble if $\lambda = n!/n^n$. By applying the same arguments to $\lambda_1 |x_1| + \ldots + \lambda_n |x_n|$, where

$\lambda_1, \ldots, \lambda_n$ are positive parameters whose product is 1, one can deduce that (3) then has in fact infinitely many solutions. Further, by suitable choice of $\lambda_1, \ldots, \lambda_n$, it follows that there are then infinitely many solutions of (3) for which any $n-1$ of $|x_1|, \ldots, |x_n|$ are arbitrarily small (possibly zero).

But in most of the cases where one has reason to suppose that there are always an infinity of solutions of an inequality such as (2), it is not easy to establish this by an extension of the method used to prove the existence of a single solution. Especially is this true of the more delicate types of argument which have been evolved to prove the solubility of certain inequalities with the best possible values for the constants.

Much light has been thrown on the question by the recent work of Mahler (1946*a*, *b*) on lattice points in n-dimensional star bodies. The following result, implicit in his work (see Mahler 1946*a*, theorem 23), is one of the most interesting from our present point of view. Suppose the inequality

$$|F(x_1, \ldots, x_n)| \leqslant \lambda \tag{4}$$

is soluble in $u_1, \ldots, u_n$ for every set $x_1, \ldots, x_n$ of linear forms in $u_1, \ldots, u_n$ with real coefficients and determinant 1. Then it may be proved by Mahler's methods (see § 2 below) that, under certain general conditions on F, the inequality

$$|F(x_1, \ldots, x_n)| \leqslant \lambda' \tag{5}$$

has infinitely many solutions, for every $\lambda' > \lambda$. The conditions in question are that F possesses automorphisms, i.e. linear transformations with real coefficients which leave $|F|$ invariant, such that (i) every point $(x_1, \ldots, x_n)$ for which $|F| \leqslant 1$ can be transformed, by an appropriate automorphism, into a point in a bounded region, and (ii) every point except the origin O can be transformed, by appropriate automorphisms, into points arbitrarily far from O.

Among the functions which satisfy these conditions are†

$$F_1(x_1, \ldots, x_n) = x_1 \ldots x_n \qquad (h = n), \tag{6}$$

$$F_2(x_1, \ldots, x_n) = x_1 \ldots x_r(x_{r+1}^2 + x_{r+2}^2) \ldots (x_{n-1}^2 + x_n^2) \qquad (h = n), \tag{7}$$

$$F_3(x_1, \ldots, x_n) = x_1^2 \ldots x_r^2(x_{r+1}^2 + \ldots + x_n^2)^{n-r} \qquad (h = 2n), \tag{8}$$

$$F_4(x_1, \ldots, x_n) = x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_n^2 \qquad (h = 2). \tag{9}$$

Here $1 \leqslant r \leqslant n-1$, except in (7), where $n \geqslant 3$, $0 \leqslant r \leqslant n-2$, and $n-r$ is even.

Our object is to carry further the work of Mahler in this direction, and to prove some additional results concerning the number and the distribution of the solutions of various Diophantine inequalities. In a later paper, we hope to consider similar questions for some particular non-homogeneous inequalities.

2. Star bodies

We first summarize briefly those of Mahler's definitions and results which are relevant to our purpose. The inequality

$$|F(x_1, \ldots, x_n)| \leqslant 1 \tag{10}$$

defines a *star body* K in n-dimensional space. A lattice is said to be admissible for K if none of its points, except O, is strictly inside K. We shall consider only bodies for which admissible

† In this connexion, see Mahler (1946*a*, § 15).

lattices exist, i.e. bodies of the *finite type*. The lower bound of the determinants of all lattices that are admissible for K is denoted by $\Delta(K)$. The arithmetical interpretation of $\Delta(K)$ is as follows: *the lower bound* $M(F)$ *of the numbers* λ *for which* (4) *is soluble for every set of linear forms of determinant* 1 *is*

$$M(F) = \{\Delta(K)\}^{-h/n}. \tag{11}$$

For to assert the insolubility of (4), with strict inequality, for some set of linear forms, is the same as to assert the existence of an admissible lattice for K of determinant $\lambda^{-n/h}$; and in defining the lower bound of the numbers λ it is immaterial whether or not we admit equality in (4).

Mahler (1946*a*, theorem 8) established the existence, for every star body of the finite type, of one or more *critical* lattices, that is, admissible lattices of determinant $\Delta(K)$. Expressed arithmetically, this means that *there exists at least one set of linear forms of determinant* 1 *for which* (4) *is not soluble with strict inequality when* $\lambda = M(F)$.

An *automorphic* star body is one for which F has automorphisms satisfying the condition (i) of § 1, so that every point of K can be transformed, by an appropriate automorphism, into a point of a bounded region. Mahler (1946*a*, theorem 21) proved that every automorphic star body possesses a critical lattice which has at least one point on the boundary of K. Expressed arithmetically this means that *there are linear forms of determinant* 1 *such that* (4) *is soluble with equality, but not with strict inequality, when* $\lambda = M(F)$.

The function F, or the body K, is said to be *fully automorphic* when its automorphisms satisfy both the conditions (i) and (ii) of § 1. Mahler (1946*a*, theorem 23) proved that every critical lattice of a fully automorphic star body has, for any $\epsilon > 0$, an infinity of points satisfying

$$1 \leqslant |F(x_1, \ldots, x_n)| < 1 + \epsilon. \tag{12}$$

We shall show (see lemma 1 and theorem 1 below), by a slight modification of his proof, that *every* lattice whose determinant does not exceed $\Delta(K)$ has, for any $\epsilon > 0$, an infinity of points satisfying

$$|F(x_1, \ldots, x_n)| < 1 + \epsilon. \tag{13}$$

If we define $\Delta_\infty(K)$ to be the lower bound of the determinants of all lattices which have only a finite number of points inside K, it follows that

$$\Delta_\infty(K) = \Delta(K) \tag{14}$$

for any fully automorphic star body. Similarly, we define $M_\infty(F)$ to be the lower bound of the numbers λ such that

$$|F(x_1, \ldots, x_n)| \leqslant \lambda$$

has infinitely many solutions for all linear forms $x_1, \ldots, x_n$ of determinant 1. *Then*

$$M_\infty(F) = M(F)$$

if F *is fully automorphic*. This is the result stated in § 1.

In order to obtain more precise results it is necessary to use the notion of reducibility, introduced in Mahler (1946*b*). A star body K is said to be *boundedly reducible* if there exists a bounded star body H contained in K such that $\Delta(H) = \Delta(K)$. Mahler proves† that the bodies defined by F_1 when $n = 2$ or 3, by F_2 when $r = 1$ and $n = 3$ and by F_4 when $n = 2$ or 3 or 4, are all boundedly reducible. The proof depends in each case on previous knowledge

† Mahler (1946*b*, §§ 13, 14, 15, 16); we shall see, however, that the proof in § 15 needs some modification.

of the value of $\Delta(K)$, and on some previous knowledge concerning the critical lattices of K. Another example of a boundedly reducible star body is provided by†

$$-1 \leqslant x_1 x_2 \leqslant k, \tag{15}$$

where k is any positive integer. Segre (1945) found the critical determinant of this body, and it follows from his proof, or more easily from the work of Cassels (1947), that this body is boundedly reducible.

A slightly different definition, in place of that of bounded reducibility, seems to be more appropriate to our problem, and allows us to obtain more precise results. We shall say that K is *fully reducible* if there exists a bounded star body H contained in K such that (i) $\Delta(H) = \Delta(K)$, and (ii) H has exactly the same critical lattices as K. If K is fully reducible, it is certainly boundedly reducible. We shall see in the next section that the boundedly reducible bodies specified in the last paragraph are all fully reducible. It is easily seen that the definition may be expressed in the alternative form: K is fully reducible if it contains a bounded star body H such that every lattice, whose determinant does not exceed $\Delta(K)$, which is admissible for H is also admissible for K.

Before we state our first general theorem it is convenient to prove three lemmas.

LEMMA 1. *Let K be a fully automorphic star body. Let k be a positive integer, and λ a positive number less than* 1. *Then there exists a bounded star body $H_{k,\lambda}$ contained in K such that any lattice Λ with $d(\Lambda) \leqslant \lambda\Delta(K)$ has at least $2k+1$ points in $H_{k,\lambda}$.*

Proof. Let $K^{(t)}$ denote, for any $t > 0$, the bounded star body consisting of those points of K whose distance from O does not exceed t.

By a theorem of Mahler (1946*a*, theorem 10), we have

$$\lim_{t\to\infty} \Delta(K^{(t)}) = \Delta(K).$$

Hence we can choose t so that $\Delta(K^{(t)}) > \lambda\Delta(K)$.

The lattice Λ will then have a point P other than O in $K^{(t)}$ and so will have at least three points in $K^{(t)}$, namely, O, P and the image of P in O. Taking $H_{1,\lambda}$ to be $K^{(t)}$, we see that the conclusion holds when $k = 1$.

We now proceed by induction on k, keeping λ fixed. Suppose the desired body $H_{k,\lambda}$ exists for some k, and denote it simply by H. Suppose there is no such body for $k+1$. Then, for every bounded star body contained in K there will be a lattice, whose determinant does not exceed $\lambda\Delta(K)$, which has at most $2k+1$ points in that body. In particular, taking the body to be $K^{(r)}$, where $r = 1, 2, \ldots$, there will be a lattice $\Lambda^{(r)}$ with $d(\Lambda^{(r)}) \leqslant \lambda\Delta(K)$ such that $\Lambda^{(r)}$ has at most $2k+1$ points in $K^{(r)}$. By the hypothesis of the induction, however, $\Lambda^{(r)}$ has at least $2k+1$ points in H. Consequently, if r is sufficiently large, $\Lambda^{(r)}$ has exactly $2k+1$ points in $K^{(r)}$, and they all lie in H. Denote these points by $P_1^{(r)}, \ldots, P_{2k+1}^{(r)}$.

Since the lattices $\Lambda^{(r)}$ have bounded determinants, and each has only a bounded number of points in H, it follows (by Mahler 1946*a*, theorem 2 and its proof) that this sequence of lattices contains a convergent subsequence $\Lambda^{(r_1)}, \Lambda^{(r_2)}, \ldots$ such that also each sequence of points $P_1^{(r_1)}, P_1^{(r_2)}, \ldots$, etc., is convergent. Denote the limiting lattice by Λ, and the limiting points by $P_1, \ldots, P_{2k+1}$, where P_1 is not O. The lattice $\Lambda^{(r_i)}$ has exactly $2k+1$ points in $K^{(r_i)}$, and these are arbitrarily near to $P_1, \ldots, P_{2k+1}$.

† This body corresponds to $F(x_1, x_2) = \max(-x_1 x_2, x_1 x_2/k)$.

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We now use the fact that K has automorphisms which transform any point other than O into points arbitrarily far from O. Let Ω be a particular automorphism of K such that the point ΩP_1 is outside H. Then, provided r_i is sufficiently large, the lattice $\Omega\Lambda^{(r_i)}$ has a point $\Omega P_1^{(r_i)}$ outside H, and also, by the hypothesis of the induction, has at least $2k+1$ points in H. But now the lattice $\Lambda^{(r_i)}$ has at least $2k+1$ points in the bounded body $\Omega^{-1}H$, and has two additional points $\pm P^{(r_i)}$ in $K^{(r_i)}$. If r_i is sufficiently large, $K^{(r_i)}$ contains $\Omega^{-1}H$, and so $\Lambda^{(r_i)}$ has at least $2k+3$ points in $K^{(r_i)}$, contrary to supposition. This contradiction establishes the induction from k to $k+1$, and so completes the proof of lemma 1.

Lemma 2. *Suppose that K is a fully automorphic and boundedly reducible star body, and let k be a positive integer. Then there is a bounded star body H_k contained in K such that, if Λ is a lattice with determinant $\Delta \leqslant \Delta(K)$, then there are at least $2k+1$ points of Λ in H_k.*

Proof. The result is true when $k=1$ by the definition of a boundedly reducible star body. The general result follows by induction just as in the proof of lemma 1; in fact, the proof may be used word for word if λ is replaced by 1.

Lemma 3. *Suppose that K is a fully automorphic and fully reducible star body, and let k be a positive integer. Then there is a bounded star body H_k^* contained in K such that, if Λ is any lattice with determinant $\Delta \leqslant \Delta(K)$, then there are at least $2k+1$ points of Λ in H_k^*; at least $2k+1$ being strictly inside H_k^* unless Λ is a critical lattice of K.*

Proof. We first prove that there exists a bounded star body H_k' contained in K such that, if Λ is any lattice, which is not a critical lattice of K, with determinant $\Delta \leqslant \Delta(K)$, then there are at least $2k+1$ points of Λ strictly inside H_k'. This is true when $k=1$ by the definition of a fully reducible star body. The result follows by induction just as in the proof of lemma 1. It is necessary to make a number of minor changes in the proof; λ is taken to be 1 throughout, the lattices $\Lambda^{(r)}$ (but not necessarily Λ) are chosen to be lattices, which are not critical lattices of K, with determinants less than or equal to $\Delta(K)$, and we are concerned only with points of these lattices $\Lambda^{(r)}$ which are strictly inside the various bodies.

Next, since K is boundedly reducible, it follows by lemma 2 that there is a bounded star body H_k contained in K such that, if Λ is any lattice with determinant $\Delta \leqslant \Delta(K)$, then there are at least $2k+1$ points of Λ in H_k.

Now it is clear that the conclusion of the lemma is valid for any bounded star body H_k^* contained in K and containing the bounded star bodies H_k and H_k'. This completes the proof of the lemma.

We now state and prove

Theorem 1. *Suppose the body K defined by*

$$|F(x_1, \ldots, x_n)| \leqslant 1 \tag{16}$$

is fully automorphic, and suppose that Λ is a lattice with determinant Δ. Then

(*a*) *if $\Delta < \Delta(K)$, there are an infinity of solutions of* (16) *with strict inequality;*

(*b*) *if K is boundedly reducible and $\Delta \leqslant \Delta(K)$, there are an infinity of solutions of* (16); *and*

(*c*) *if K is fully reducible and $\Delta \leqslant \Delta(K)$, there are an infinity of solutions of* (16) *with strict inequality, unless Λ is a critical lattice of K, in which case there are an infinity of solutions of* (16) *with equality.*

Proof. The conclusion (*a*) is merely a restatement of the result we considered at the beginning of this section and which, as we said, can be deduced from a modified form of one of Mahler's proofs. This conclusion is also an immediate consequence of lemma 1.

The conclusions (*b*) and (*c*) follow immediately from lemmas 2 and 3 respectively.

It will be noted that this theorem merely proves (in certain circumstances) the *existence* of an infinity of solutions of the inequality (16); it gives no information about the distribution of the solutions throughout the different parts of the body. In particular cases, more information can be obtained by direct application of lemmas 1, 2 or 3. We return to this point in § 4.

3. Some fully reducible star bodies

All the star bodies which we mentioned on p. 313 as being boundedly reducible are in fact fully reducible. The method which will be used to prove this is a straightforward modification of that of Mahler, and rests on modified forms of theorems K and L, and definition D of Mahler (1946*b*). We denote these modified forms by accented letters. As before, we use $K^{(t)}$ to denote, for any $t>0$, the bounded star body consisting of those points of K whose distance from O does not exceed t.

Theorem K′. *Suppose the star body K is not fully reducible. Then there exists a critical lattice Λ of K and an infinite sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ such that*

(*a*) *Λ_r is admissible for $K^{(r)}$, but not for K;*

(*b*) $d(\Lambda_r) \leqslant \Delta(K)$;

(*c*) *$\Lambda_r \to \Lambda$ as $r \to \infty$.*

Proof. Since $K^{(r)}$ is a bounded star body contained in K, and K is not fully reducible, there exists a critical lattice $\Lambda^{(r)}$ of $K^{(r)}$ which is not admissible for K. Since

$$d(\Lambda^{(r)}) = \Delta(K^{(r)}) \leqslant \Delta(K), \tag{17}$$

and $\Lambda^{(r)}$ is admissible for $K^{(r)}$, the lattices $\Lambda^{(1)}, \Lambda^{(2)}, \ldots$ form a bounded sequence; and so, by theorem 2 of Mahler (1946*a*), they contain a subsequence

$$\Lambda_1 = \Lambda^{(k_1)}, \quad \Lambda_2 = \Lambda^{(k_2)}, \quad \ldots,$$

converging to some limiting lattice, Λ, say.

Since $k_r \geqslant r$, the lattice $\Lambda_r = \Lambda^{(k_r)}$ is admissible for $K^{(r)}$ but not for K, and consequently (*a*) holds. Conclusion (*b*) follows from (17) and conclusion (*c*) is obvious. We have only to prove that Λ is critical for K. Since the lattices Λ_r converge to the lattice Λ, we have

$$d(\Lambda) = \lim_{r\to\infty} d(\Lambda_r),$$

and by (17) and the corollary to theorem 10 of Mahler (1946*a*),

$$\lim_{r\to\infty} d(\Lambda_r) = \lim_{r\to\infty} d(\Lambda^{(r)}) = \lim_{r\to\infty} \Delta(K^{(r)}) = \Delta(K).$$

Thus $d(\Lambda) = \Delta(K)$, and we have only to prove that Λ has no point, other than O, strictly inside K. If P were such a point, then P would be strictly inside $K^{(r)}$ for all large r. But the lattice Λ_r has a point P_r which tends to P as $r \to \infty$, and this contradicts (*a*).

Definition D′. *A critical lattice Λ of K is said to be fully critical if there exists a bounded star body H contained in K with the following property: any lattice Λ' which is sufficiently near to Λ and is admissible for H either has $d(\Lambda') > d(\Lambda)$ or is a critical lattice for K.*

Theorem L′. *Suppose that every critical lattice of K is fully critical. Then K is fully reducible.*

Proof.† Suppose K is not fully reducible. By theorem K′, there exists a critical lattice Λ of K and an infinite sequence $\Lambda_1, \Lambda_2, \ldots$ of lattices for which (*a*), (*b*) and (*c*) hold. We assert that Λ cannot be a fully critical lattice of K. For if H is any bounded part of K, then H will be contained in $K^{(r)}$ for all large r, and Λ_r will be arbitrarily near to Λ and will be admissible for H. Now (*b*) asserts that $d(\Lambda_r) \leqslant d(\Lambda)$, and (*a*) asserts that Λ_r is not a critical lattice of K. Thus, by definition D′, Λ is not a fully critical lattice of K. This contradiction proves the theorem.

In all those cases, specified on p. 313, in which Mahler proved K to be boundedly reducible, the same argument, with theorem L′ in place of theorem L, shows that K is also fully reducible. Mahler's proof is, in each case, based on the fact that every critical lattice of K is of the form $\Omega\Lambda_0$, where Ω is an automorphism of K and Λ_0 is a certain special critical lattice. His argument proves the existence of a bounded star body K^* contained in K with the property that every lattice Λ^*, which is admissible for K^* and which is sufficiently near to Λ_0, is either a critical lattice of K or has determinant greater than $\Delta(K)$. Thus Λ_0 is fully critical, and so every critical lattice of K is fully critical; consequently it follows from theorem L′ that K is fully reducible.

A further investigation is needed, however, in the case of the three-dimensional star body K defined by

$$|(x_1^2+x_2^2)\,x_3| \leqslant 1, \tag{18}$$

since Mahler's proof that this body is boundedly reducible is incomplete. It was proved by Davenport (1939) that

$$\Delta(K) = \tfrac{1}{2}\sqrt{(23)}. \tag{19}$$

One known critical lattice of K is the lattice Λ_0 given by

$$x_1+ix_2 = u_1+\theta u_2+\theta^2 u_3, \quad x_3 = u_1+\phi u_2+\phi^2 u_3, \tag{20}$$

where θ is the complex root and ϕ is the real root of the cubic equation $t^3-t-1=0$. Automorphisms Ω of K are given by

$$x_1+ix_2 = \lambda(x_1'\pm ix_2'), \quad x_3 = \mu x_3', \tag{21}$$

where λ is any complex number and μ is any real number, subject to $|\lambda^2\mu| = 1$. Thus every lattice of the form $\Omega\Lambda_0$ is also a critical lattice of K. Mahler bases his proof that K is boundedly reducible on the statement (attributed to Mordell) that *every* critical lattice of K is of this form. But Mordell (1942) did not prove as much as this; his results were as follows. (*a*) If L_1, L_2, L_3 are real linear forms with determinant $\Delta \neq 0$, the inequality

$$|(L_1^2+L_2^2)\,L_3| < \frac{2}{\sqrt{(23)}}\,|\Delta| + \epsilon$$

† The proof of theorem L in Mahler (1946*b*) does not seem to be worded quite clearly. The body K^* determines t, and t determines Λ. By definition D, the lattice Λ, if it is strongly critical, determines another bounded body K' contained in K. It is plain that K' cannot legitimately be confused with K^*. However, no difficulty arises if clause (*a*) of theorem K is read with $K^{(r+t)}$ in place of $K^{(t)}$.

is soluble for any $\epsilon > 0$ with integral values, not all zero, of the variables. (*b*) The least value of the product on the left is $2|\Delta|/\sqrt{(23)}$ when and only when

$$(L_1^2+L_2^2)\,L_3 \sim \frac{2}{\sqrt{(23)}}\,|\Delta|\,\Pi(u_1+\theta u_2+\theta^2 u_3). \tag{22}$$

The statement (*a*) implies that $\Delta(K) \geqslant \frac{1}{2}\sqrt{(23)}$. The statement (*b*), read with the word 'when', tells us that the particular lattice given by (20) is admissible, and it follows that $\Delta(K) = \frac{1}{2}\sqrt{(23)}$. The other half of the statement (*b*) asserts that if the product attains its lower bound, and the value of this lower bound is $2|\Delta|/\sqrt{(23)}$, then the product satisfies (22). The hypothesis is satisfied by the product corresponding to a critical lattice of K provided that this critical lattice has a point on the boundary of K. It follows that every critical lattice of K, *which has a lattice point on the boundary of* K, is of the form $\Omega\Lambda_0$ for some automorphism Ω of K.

It is, however, possible to modify Mahler's proof by basing it on a weaker proposition, which can be deduced from the result Mordell did prove by the method of the proof of Mahler (1946*a*, theorem 20). This proposition, which can also be deduced directly either from Davenport's original proof of (19) (Davenport 1939) or from Mordell's proof (Mordell 1942), is:

LEMMA 4. *Every critical lattice of the body K defined by* (18) *is of the form* $\Omega\Lambda^*$, *where* Ω *is an automorphism of K and* Λ^* *is arbitrarily near to* Λ_0 *and has a point of the form* $(a, 0, a)$, *where a is arbitrarily near to* 1.

Proof. Suppose Λ is a critical lattice of K. There exists a sequence of points $P^{(r)}$ of Λ (not necessarily distinct) satisfying

$$1 \leqslant |(x_1^2+x_2^2)\,x_3| < 1+\frac{1}{r},$$

for $r = 1, 2, \ldots$. We can choose a sequence of automorphisms $\Omega^{(r)}$ of K such that the points $\Omega^{(r)}P^{(r)}$ are of the form $(a^{(r)}, 0, a^{(r)})$, where

$$1 \leqslant (a^{(r)})^3 < 1+\frac{1}{r}$$

for $r = 1, 2, \ldots$. The sequence of lattices $\Omega^{(r)}\Lambda$ is a bounded sequence, and so (by Mahler 1946*a*, theorem 2) contains a subsequence

$$\Omega_1\Lambda = \Omega^{(k_1)}\Lambda, \quad \Omega_2\Lambda = \Omega^{(k_2)}\Lambda, \quad \ldots$$

converging to some limiting lattice Λ', say. Since $\Omega_r\Lambda$ is a critical lattice of K, and there is a point $\Omega_r P_r$ of $\Omega_r\Lambda$ of the form $(a_r, 0, a_r)$, where

$$1 \leqslant (a_r)^3 < 1+\frac{1}{r},$$

it follows that Λ' is a critical lattice of K and that $(1, 0, 1)$ is a point of Λ' on the boundary of K.

By Mordell's result, stated earlier, the lattice Λ' must be of the form† $\Theta\Lambda_0$, where Θ is an automorphism of K. Now the lattices $\Theta^{-1}\Omega_r\Lambda$ converge to the lattice Λ_0. As $(1, 0, 1)$ is

† As $(1, 0, 1)$ is a point of Λ' it is natural to suppose that $\Lambda' = \Lambda_0$ or $\Lambda' = \overline{\Lambda}_0$, where $\overline{\Lambda}_0$ is obtained from Λ_0 by changing x_2 into $-x_2$. This is not obvious, but it is in fact true, as we prove on p. 337. The difference between the statements that every critical lattice of K is of the form $\Theta\Lambda_0$ for some automorphism Θ of K and that a critical lattice with $(1, 0, 1)$ as a lattice point is necessarily either Λ_0 or $\overline{\Lambda}_0$ seems to have escaped notice. A similar problem arises on p. 333 in connexion with the body $|x_1x_2x_3| \leqslant 1$.

a point of Λ_0, there are points Q_r of $\Theta^{-1}\Omega_r\Lambda$, for $r = 1, 2, \ldots$, converging to the point $(1, 0, 1)$. Consequently there is a sequence of automorphisms Θ_r, $r = 1, 2, \ldots$, converging to the unit automorphism and transforming the points Q_r into points of the form $(b_r, 0, b_r)$, where b_r converges to the limit 1. Taking

$$\Lambda^* = \Theta_r \Theta^{-1}\Omega_r \Lambda, \quad \Omega = \Omega_r^{-1}\Theta\Theta_r^{-1},$$

and r sufficiently large, Λ is of the form $\Omega\Lambda^*$, where Λ^* is arbitrarily near to Λ_0 and has a point of the form $(a, 0, a)$, a being arbitrarily near to 1. This proves the lemma.

We now apply this lemma to complete the proof that K is fully reducible. Mahler proves (1946*b*, pp. 623–626) that there exists a bounded star body K^* contained in K such that if Λ^* is any lattice which is sufficiently near to Λ_0, and is admissible for K^*, and has a point of the form $(a, 0, a)$ with a sufficiently near to 1, then either $d(\Lambda^*) > \Delta(K)$ or $\Lambda^* = \Lambda_0$ (the result is not stated in exactly this form, but this is the conclusion reached on p. 626). By lemma 4, any critical lattice of K is of the form $\Omega\Lambda^*$, where Λ^* satisfies the above conditions and where $d(\Lambda^*) = \Delta(K)$. Thus Λ^* must coincide with Λ_0, and so every critical lattice is of the form $\Omega\Lambda_0$. This justifies the statement which was previously unproved. The result of Mahler, just quoted, shows that Λ_0 is fully critical, and consequently every critical lattice of K is fully critical and K is fully reducible.

We conclude by listing formally those star bodies which we have shown to be fully reducible.

THEOREM 2. *The star bodies defined by the following inequalities are all fully reducible, and have the determinants stated:*

$$|x_1 x_2| \leqslant 1, \quad \Delta = \sqrt{5}; \tag{23}$$

$$-1 \leqslant x_1 x_2 \leqslant k, \quad \Delta = \sqrt{(k^2+4k)}, \tag{24}$$

for any positive integer k;

$$|x_1 x_2 x_3| \leqslant 1, \quad \Delta = 7; \tag{25}$$

$$|(x_1^2+x_2^2)\, x_3| \leqslant 1, \quad \Delta = \tfrac{1}{2}\sqrt{(23)}; \tag{26}$$

$$|x_1^2+x_2^2-x_3^2| \leqslant 1, \quad \Delta = \sqrt{\tfrac{3}{2}}; \tag{27}$$

$$|x_1^2+x_2^2+x_3^2-x_4^2| \leqslant 1, \quad \Delta = \sqrt{\tfrac{7}{4}}; \tag{28}$$

$$|x_1^2+x_2^2-x_3^2-x_4^2| \leqslant 1, \quad \Delta = \tfrac{3}{2}. \tag{29}$$

We now restate, in arithmetical form, the implications of theorem 1 for each of these bodies, as it may be convenient to have the results on record.

THEOREM 3. *Let $x_1, \ldots, x_n$ be real linear forms in $u_1, \ldots, u_n$ of determinant $D \neq 0$. Each of the following inequalities has an infinity of solutions, and, indeed, has an infinity of solutions with strict inequality unless the form in question is equivalent† to a multiple of the special form indicated. In that case there are no solutions with strict inequality other than the trivial solution in which all the variables are zero:*

$$|x_1 x_2| \leqslant \frac{1}{\sqrt{5}}|D| \qquad (n=2), \tag{30}$$

$$u_1^2 + u_1 u_2 - u_2^2; \tag{30a}$$

$$-(k^2+4k)^{-\frac{1}{2}}|D| \leqslant x_1 x_2 \leqslant k(k^2+4k)^{-\frac{1}{2}}|D| \quad (n=2), \tag{31}$$

$$k u_1^2 + k u_1 u_2 - u_2^2, \tag{31a}$$

† *Equivalence* refers to linear substitutions on $u_1, \ldots, u_n$ with integral coefficients and determinant ± 1.

where in (31) *and* (31 *a*) *k is a positive integer;*

$$|x_1 x_2 x_3| \leqslant \tfrac{1}{7}|D| \qquad (n=3), \tag{32}$$

$$u_1^3+u_2^3+u_3^3-4(u_2u_3^2+u_3u_1^2+u_1u_2^2)+3(u_2^2u_3+u_3^2u_1+u_1^2u_2)-u_1u_2u_3; \tag{32a}$$

$$|(x_1^2+x_2^2)x_3| \leqslant \frac{2}{\sqrt{(23)}}|D| \qquad (n=3), \tag{33}$$

$$u_1^3+u_2^3+u_3^3+u_3^2u_1-u_3^2u_2+2u_1^2u_3-u_2^2u_1-3u_1u_2u_3; \tag{33a}$$

$$|x_1^2+x_2^2-x_3^2| \leqslant (\tfrac{2}{3}D^2)^{\frac{1}{3}} \qquad (n=3), \tag{34}$$

$$u_1^2+u_1u_2+u_2^2-2u_3^2; \tag{34a}$$

$$|x_1^2+x_2^2+x_3^2-x_4^2| \leqslant (\tfrac{4}{7}D^2)^{\frac{1}{4}} \qquad (n=4), \tag{35}$$

$$u_1^2+u_2^2+u_3^2-u_4^2-u_1u_4-u_2u_4-u_3u_4; \tag{35a}$$

$$|x_1^2+x_2^2-x_3^2-x_4^2| \leqslant (\tfrac{4}{9}D^2)^{\frac{1}{4}} \qquad (n=4), \tag{36}$$

$$u_1^2-u_2^2-u_3^2+u_4^2+2u_1u_2+2u_1u_3+u_1u_4+u_2u_4+u_3u_4. \tag{36a}$$

The special form in each case is that which arises from the general critical lattice of the body in question. For proofs that this is so, we must refer to Mahler (1946 *b*); also, for (31), to Segre (1945), and for (34), (35) and (36) to Dickson (1930, chapters 8 and 9).

4. The distribution of the lattice points in a body

So far we have been concerned with the existence of an infinity of solutions of certain inequalities, or with the existence of an infinity of lattice points in certain bodies. We now investigate the possible distributions of the lattice points within the bodies.

A star body K_0, contained in a star body K, will be said to *generate* K, if for every bounded part K' of K there is an automorphism Ω of K such that ΩK_0 contains K'. If K_0 generates K, then obviously ΩK_0 also generates K, for every automorphisn Ω of K. A simple example to illustrate the definition is

K: $\qquad |x_1 \ldots x_n| \leqslant 1,$

K_0: $\qquad |x_1 \ldots x_n| \leqslant 1, \quad |x_2| \leqslant 1, \quad \ldots, \quad |x_n| \leqslant 1.$

For, given any bounded part K' of K, we can choose λ so large that the automorphism

$$x_1' = \lambda^{1-n}x_1, \quad x_2' = \lambda x_2, \quad \ldots, \quad x_n' = \lambda x_n$$

transforms K_0 into a body which contains K'.

Using this definition we can prove a stronger form of theorem 1.

Theorem 4. *Suppose that K is a fully automorphic star body and K_0 is a star body which generates K. Suppose Λ is a lattice with determinant Δ. Then*

(*a*) *if $\Delta < \Delta(K)$, there are an infinity of points of Λ strictly inside K_0,*

(*b*) *if K is boundedly reducible and $\Delta \leqslant \Delta(K)$, there are an infinity of points of Λ in K_0, and*

(*c*) *if K is fully reducible and $\Delta \leqslant \Delta(K)$, there are an infinity of points of Λ strictly inside K_0, unless Λ is a critical lattice of K, in which case there are an infinity of points of Λ on the boundary of K_0.*

Proof. To prove clause (*a*) we use lemma 1. We suppose that $\Delta < \Delta(K)$ and choose $\lambda < 1$ so that $\Delta < \lambda\Delta(K)$. Let k be a positive integer and consider the bounded star body $H_{k,\lambda}$ contained in K given by lemma 1. Then as K_0 generates K, there is an automorphism Ω_k of

K such that $\Omega_k K_0$ contains $H_{k,\lambda}$. Now since the determinant of the lattice $\Omega_k \Lambda$ is $\Delta < \lambda\Delta(K)$, there are at least $2k+1$ points of $\Omega_k\Lambda$ which are strictly inside $H_{k,\lambda}$, and which are therefore strictly inside $\Omega_k K_0$. Thus there are at least $2k+1$ points of Λ strictly inside K_0, and as k may be arbitrarily large there are an infinity of points of Λ in K_0.

The clauses (*b*) and (*c*) follow from lemmas 2 and 3 by the same argument.

5. The application of theorem 4 to certain Diophantine inequalities

We now apply theorem 4 to some of the bodies listed in theorem 2, and formulate the results in arithmetical terms.

Theorem 5. *Let x_1, x_2, x_3 be real linear forms in u_1, u_2, u_3 of determinant $D \neq 0$. Suppose that the product $x_1x_2x_3$ is not equivalent to a multiple of*

$$u_1^3+u_2^3+u_3^3-4(u_2u_3^2+u_3u_1^2+u_1u_2^2)+3(u_2^2u_3+u_3^2u_1+u_1^2u_2)-u_1u_2u_3. \tag{37}$$

Then the inequalities
$$|x_1x_2x_3| < \tfrac{1}{7}|D|, \quad |x_2|<\epsilon, \quad |x_3|<\epsilon \tag{38}$$
have infinitely many solutions for any $\epsilon>0$. If $x_1x_2x_3$ is equivalent to a multiple of (37), *there are no solutions of* (38) *other than $u_1=u_2=u_3=0$, but there are infinitely many solutions of*

$$|x_1x_2x_3| = \tfrac{1}{7}|D|, \quad |x_2|<\epsilon, \quad |x_3|<\epsilon \tag{39}$$

for any $\epsilon>0$.

Proof. The body K defined by $|x_1x_2x_3| \leqslant \frac{1}{7}|D|$ is fully automorphic and fully reducible, and is generated by the body K_0 defined by

$$|x_1x_2x_3| \leqslant \tfrac{1}{7}|D|, \quad |x_2|\leqslant\epsilon, \quad |x_3|\leqslant\epsilon.$$

Also $\Delta(K) = |D|$.

The lattice of points (x_1, x_2, x_3) corresponding to integral values of u_1, u_2, u_3 has determinant $|D|$, and is a critical lattice of K if and only if the product $x_1x_2x_3$ is equivalent to a multiple of the form (37). The conclusions now follow from (*c*) of theorem 4. The assertion concerning (39) may also be deduced from classical results concerning the units of totally real cubic fields, since the form (37) is the norm-form for the field $k(\theta)$, where $\theta^3+\theta^2-2\theta-1=0$.

Theorem 6. *Let x_1, x_2, x_3 be real linear forms in u_1, u_2, u_3 of determinant $D\neq 0$. Suppose that the product $(x_1^2+x_2^2)\,x_3$ is not equivalent to a multiple of*

$$u_1^3+u_2^3+u_3^3+u_3^2u_1-u_3^2u_2+2u_1^2u_3-u_2^2u_1-3u_1u_2u_3. \tag{40}$$

Then the inequalities
$$|(x_1^2+x_2^2)\,x_3| < \frac{2}{\sqrt{(23)}}|D|, \quad |x_3|<\epsilon \tag{41}$$

have infinitely many solutions for any $\epsilon>0$, as also have the inequalities

$$|(x_1^2+x_2^2)\,x_3| < \frac{2}{\sqrt{(23)}}|D|, \quad x_1^2+x_2^2<\epsilon. \tag{42}$$

If $(x_1^2+x_2^2)\,x_3$ is equivalent to a multiple of (40), *then* (41) *and* (42) *have an infinity of solutions if the first inequality signs are replaced by equalities.*

Proof. This is similar to that of theorem 5, and depends on the facts that the body

$$|(x_1^2+x_2^2)\,x_3| \leqslant \frac{2}{\sqrt{(23)}}|D|$$

is fully automorphic and fully reducible, and can be generated either by

$$|(x_1^2+x_2^2)\,x_3| \leqslant \frac{2}{\sqrt{(23)}}|D|, \quad |x_3| \leqslant \epsilon,$$

or by

$$|(x_1^2+x_2^2)\,x_3| \leqslant \frac{2}{\sqrt{(23)}}|D|, \quad x_1^2+x_2^2 \leqslant \epsilon.$$

Theorem 7. *Let $Q(u_1, u_2, u_3)$ be an indefinite ternary quadratic form with determinant $d \neq 0$, and let $\overline{Q}(u_1, u_2, u_3)$ denote the adjoint form. Let λ_1, λ_2, λ_3 be any real numbers satisfying*

$$\overline{Q}(\lambda_1, \lambda_2, \lambda_3) = 0. \tag{43}$$

Then the inequalities

$$|Q(u_1, u_2, u_3)| \leqslant (\tfrac{2}{3}|d|)^{\frac{1}{3}}, \quad |\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3| < \epsilon \tag{44}$$

have infinitely many solutions for any $\epsilon > 0$. If $Q(u_1, u_2, u_3)$ is not equivalent to a multiple of

$$u_1^2+u_1u_2+u_2^2-2u_3^2, \tag{45}$$

these conclusions hold with strict inequality in the first part of (44).

Proof. Without loss of generality we can suppose $d<0$. There exist linear forms x_1, x_2, x_3 in u_1, u_2, u_3, of determinant $\sqrt{\frac{3}{2}}$, such that

$$Q(u_1, u_2, u_3) = (-\tfrac{2}{3}d)^{\frac{1}{3}}\,(x_1^2+x_2^2-x_3^2)$$

identically in u_1, u_2, u_3. Then

$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 = \mu_1 x_1+\mu_2 x_2+\mu_3 x_3,$$

where

$$\mu_1^2+\mu_2^2 = \mu_3^2$$

by (43). By applying successively two linear transformations of the forms

$$x_1 = x_1'\cos\theta+x_2'\sin\theta, \quad x_2 = -x_1'\sin\theta+x_2'\cos\theta, \quad x_3 = x_3',$$

and

$$x_1 = x_1'\cosh u+x_3'\sinh u, \quad x_2 = x_2', \quad x_3 = x_1'\sinh u+x_3'\cosh u,$$

each of which leaves $x_1^2+x_2^2-x_3^2$ invariant, we can ensure that

$$\mu_1 = 1, \quad \mu_2 = 0, \quad \mu_3 = 1,$$

so that

$$\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3 = x_1+x_3.$$

The star body K defined by

$$|x_1^2+x_2^2-x_3^2| \leqslant 1$$

is fully automorphic and fully reducible, and $\Delta(K) = \sqrt{\frac{3}{2}}$. It is generated by the body K_0 defined by

$$|x_1^2+x_2^2-x_3^2| \leqslant 1, \quad |x_1+x_3| \leqslant \epsilon.$$

The lattice of points (x_1, x_2, x_3) which correspond to integral values of u_1, u_2, u_3 has determinant $\sqrt{\frac{3}{2}}$, and is a critical lattice of K if and only if $Q(u_1, u_2, u_3)$ is equivalent to a multiple of (45). The conclusions now follow from (*c*) of theorem 4.

Theorems analogous to theorem 7 hold also for indefinite quaternary quadratic forms. For a form $Q(u_1, u_2, u_3, u_4)$ of signature $(3, 1)$, an appropriate linear inequality corresponding to that in (44) is

$$|\lambda_1 u_1+\lambda_2 u_2+\lambda_3 u_3+\lambda_4 u_4| < \epsilon, \tag{46}$$

where $\overline{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0$. For a form of signature $(2, 2)$, one can impose two linear inequalities:

$$|\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4| < \epsilon, \quad |\mu_1 u_1 + \mu_2 u_2 + \mu_3 u_3 + \mu_4 u_4| < \epsilon, \tag{47}$$

where the λ's and μ's are any real numbers satisfying

$$\overline{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0, \quad \overline{Q}(\mu_1, \mu_2, \mu_3, \mu_4) = 0,$$

$$\lambda_1 \frac{\partial \overline{Q}}{\partial \mu_1} + \lambda_2 \frac{\partial \overline{Q}}{\partial \mu_2} + \lambda_3 \frac{\partial \overline{Q}}{\partial \mu_3} + \lambda_4 \frac{\partial \overline{Q}}{\partial \mu_4} = 0.$$

For it is possible to transform Q into $x_1^2 + x_2^2 - x_3^2 - x_4^2$ in such a way that the two inequalities (47) become

$$|x_1 + x_3| < \epsilon, \quad |\rho(x_1 + x_3) + x_2 + x_4| < \epsilon,$$

and the proof then follows the same lines as before.

6. A further result on the distribution of lattice points

We now prove a result applying to star bodies which, though automorphic, are not necessarily fully automorphic. This will serve as a partial substitute for theorem 4, which related only to fully automorphic star bodies.

Let K be an automorphic star body, and let K_0 generate K. We say that an automorphism Θ of K *reduces* K_0 if ΘK_0 is properly contained in K_0. We define $C(K_0, \Theta)$ to consist of all points which are common to all the star bodies $\Theta^\nu K_0$ for $\nu = 1, 2, \ldots$; and we call $C(K_0, \Theta)$ a *core* of K. Obviously the origin belongs to every core. It is clear that

$$\Theta C(K_0, \Theta) = C(K_0, \Theta).$$

Perhaps the simplest example of a star body which is automorphic, but not fully automorphic, is the hyperbolic cylinder, defined by

$$|x_1 x_2| \leqslant 1, \quad |x_3| \leqslant 1. \tag{48}$$

Here we can take K_0 to be the star body defined by

$$|x_1 x_2| \leqslant 1, \quad |x_2| \leqslant 1, \quad |x_3| \leqslant 1, \tag{49}$$

since then K_0 clearly generates K. Let Θ be the automorphism

$$x_1 = \tfrac{1}{2}x_1', \quad x_2 = 2x_2', \quad x_3 = x_3'. \tag{50}$$

Then Θ reduces K_0, since ΘK_0 is defined by

$$|x_1 x_2| \leqslant 1, \quad |x_2| \leqslant \tfrac{1}{2}, \quad |x_3| \leqslant 1.$$

Since $\Theta^\nu K_0$ is given by $\quad |x_1 x_2| \leqslant 1, \quad |x_2| \leqslant \dfrac{1}{2^\nu}, \quad |x_3| \leqslant 1,$

we see that $C(K_0, \Theta)$ is simply the infinite rectangle

$$x_2 = 0, \quad |x_3| \leqslant 1.$$

We can now state and prove the theorem which takes the place of theorem 4.

Theorem 8. *Suppose that K is a star body generated by a star body K_0 and that Θ is an automorphism of K which reduces K_0. Suppose that Λ is a lattice with determinant Δ which has no point other than O in $C(K_0, \Theta)$. Then the conclusions (a), (b) and (c) of theorem 4 are valid.*

Proof. We show first that
$$\Delta(K_0) = \Delta(K). \tag{51}$$
As K_0 generates K, there exists, for any $t>0$, an automorphism Ω of K such that ΩK_0 contains $K^{(t)}$. Now
$$\Delta(K) \geqslant \Delta(K_0) = \Delta(\Omega K_0) \geqslant \Delta(K^{(t)}).$$
But, by a theorem of Mahler (1946*a*, equation (8·21)), $\Delta(K^{(t)})$ tends to $\Delta(K)$ as $t\to\infty$. This proves (51).

We can now prove the conclusion (*a*) of theorem 4. Suppose that $\Delta<\Delta(K)$. Then since $\Delta<\Delta(K) = \Delta(\Theta^\nu K_0)$, there is a point other than O of Λ in the interior of $\Theta^\nu K_0$ for $\nu = 0, 1, 2, \ldots$. If these points comprised only a finite number of distinct points, then at least one of them would be common to all the bodies $\Theta^\nu K_0$, and so would lie in $C(K_0, \Theta)$, contrary to our hypothesis. Thus there are an infinity of points of Λ in the interior of K_0.

Suppose now that $\Delta\leqslant\Delta(K)$ and that K is boundedly reducible. Then there is a bounded star body K' contained in K with $\Delta(K') = \Delta(K)$. Since K_0 generates K there is an automorphism Ω of K such that ΩK_0 contains K'. Now the lattice $\Omega\Theta^{-\nu}\Lambda$ has determinant
$$\Delta\leqslant\Delta(K) = \Delta(K'),$$
and so there is a point other than O of $\Omega\Theta^{-\nu}\Lambda$ which is in K' and which is therefore in ΩK_0. Thus there is a point other than O of Λ in the body $\Theta^\nu K_0$, for $\nu = 0, 1, 2, \ldots$. It follows just as in the last paragraph that there are an infinity of points of Λ in K_0. This proves that the conclusion (*b*) of theorem 4 is valid in the present case.

The conclusion (*c*) of theorem 4 may be established by a very similar argument, using the fact that, if K is fully reducible, there is a bounded star body K' contained in K such that $\Delta(K') = \Delta(K)$ and such that every lattice, which has determinant $\Delta\leqslant\Delta(K)$ and which has no point other than O strictly inside K', is necessarily a critical lattice of K.

Corollary. *If $d(\Lambda)<\Delta(K)$, then either Λ has an infinity of points in the interior of K or Λ has a point other than O in every core of K.*

We conclude this section by applying the corollary to the body K defined by (48). As we have seen, one core of K is given by
$$x_2 = 0, \quad |x_3|\leqslant 1. \tag{52}$$
The only other core of K is obtained by interchanging the roles played by x_1 and x_2, and is given by
$$x_1 = 0, \quad |x_3|\leqslant 1. \tag{53}$$
It was proved by Varnavides (1948) that the critical determinant of K is $\sqrt{5}$. Hence the above corollary tells us that any lattice Λ with determinant less than $\sqrt{5}$ has an infinity of points inside the body
$$|x_1 x_2|\leqslant 1, \quad |x_3|\leqslant 1,$$
except possibly if the lattice has a point other than O satisfying (52), and also has a point other than O satisfying (53).

We can state this result in an arithmetical form as follows:

Theorem 9. *Let x_1, x_2, x_3 be real linear forms in u_1, u_2, u_3 of determinant $D\neq 0$. Let λ, μ be any positive numbers satisfying*
$$\lambda\mu>\frac{1}{\sqrt{5}}|D|. \tag{54}$$

Then the inequalities $$|x_1 x_2| < \lambda, \quad |x_3| < \mu \tag{55}$$
have an infinity of solutions, except possibly if there exist integers u_1, u_2, u_3, *not all zero, satisfying* (52) *and there also exist integers* u_1, u_2, u_3, *not all zero, satisfying* (53).

The following examples are of interest in connexion with this theorem. First, there is the example
$$x_1 = u_1 + \theta u_2, \quad x_2 = u_1 + (1-\theta) u_2, \quad x_3 = \alpha u_1 + \beta u_2 + \epsilon u_3, \tag{56}$$
where $\epsilon > 0$ and $\theta = \frac{1}{2}(1+\sqrt{5})$ and α, β are any real numbers. The determinant of the forms is $-\epsilon\sqrt{5}$, which may be arbitrarily small; but the inequalities
$$|x_1 x_2| < 1, \quad |x_3| < 1 \tag{57}$$
have only a finite number of solutions. For since
$$x_1 x_2 = u_1^2 + u_1 u_2 - u_2^2,$$
the first inequality implies that $u_1 = u_2 = 0$, and the second then allows only a finite number of possibilities for u_3.

Secondly, there is the less obvious example
$$x_1 = k(u_1 + \theta u_2 + \theta u_3), \quad x_2 = k(u_1 + (1-\theta) u_2 + \theta u_3), \quad x_3 = \epsilon u_3, \tag{58}$$
where $k > 1$ and $\epsilon > 0$. The determinant of the forms is $-\epsilon k^2\sqrt{5}$, which may be arbitrarily small, but again the inequalities (57) have only a finite number of solutions. To prove this, we assume that the inequality
$$|k^2(u_1 + \theta u_2 + \theta u_3)(u_1 + (1-\theta) u_2 + \theta u_3)| < 1 \tag{59}$$
has an infinity of solutions for which u_3 is bounded, and obtain a contradiction. We have
$$|(u_1 + \theta u_2 + \theta u_3)(u_1 + (1-\theta) u_2 + (1-\theta) u_3)| \geqslant 1,$$
unless $u_1 = u_2 + u_3 = 0$, and it follows from (59) that
$$|u_1 + (1-\theta) u_2 + \theta u_3| < \frac{1}{k^2} |u_1 + (1-\theta) u_2 + (1-\theta) u_3|.$$
Similarly
$$|u_1 + \theta u_2 + \theta u_3| < \frac{1}{k^2} |u_1 + \theta u_2 + (1-\theta) u_3|,$$
unless $u_1 + u_3 = u_2 - u_3 = 0$. But since $k > 1$ and u_3 is bounded, these inequalities imply that
$$u_1 + (1-\theta) u_2 \quad \text{and} \quad u_1 + \theta u_2$$
are both bounded, so that u_1 and u_2 are both bounded, contrary to hypothesis.

These examples are consistent with theorem 4 because K is not fully automorphic, and they are consistent with theorem 9, because, in each case, there are integers u_1, u_2, u_3, not all zero, satisfying (52), and also integers u_1, u_2, u_3, not all zero, satisfying (53).

7. Isolation theorems

Let $F(x_1, \ldots, x_n)$ be any one of the forms in theorem 3. The existing literature of the geometry of numbers contains results which not only give the best possible inequalities of the form
$$|F(x_1, \ldots, x_n)| \leqslant \lambda$$
which are always soluble, but go further, and show that this best possible inequality is, in a certain sense, 'isolated'. By this we mean that an appreciably stronger inequality is soluble

if the form in question is not equivalent to a multiple of the corresponding critical form, specified in theorem 3. Such isolation results are known for all the forms of theorem 3, except for that in (26).

Our next object is to establish some analogous results when the inequality in question is required not only to be soluble, but to have infinitely many solutions.

The simplest form, and the one about which most is known, is the form $x_1 x_2$ in two variables u_1, u_2, and an account of this case will serve as an introduction to the later work.

As we shall not, henceforward, be concerned with forms in more than four variables, we abandon the suffix notation, and denote the forms by x, y, ... or X, Y, ... and the variables by u, v, ...

The product XY of two real linear forms in u, v of determinant D is an indefinite binary quadratic form:

$$XY = au^2+buv+cv^2 = Q(u,v),$$

whose discriminant is $d = D^2$. A famous theorem of Markoff† asserts the existence of an infinite sequence Q_1, Q_2, ... of special forms, with discriminants d_1, d_2, ..., with the following properties. The inequality

$$|Q(u,v)| \leqslant \sqrt{\frac{d}{d_n}} \tag{60}$$

is soluble for any form Q which is not a multiple of one of Q_1, Q_2, ..., Q_{n-1}. The inequality (60) is soluble with equality, but not with strict inequality, if Q is equivalent to a multiple of Q_n. The numbers d_1, d_2, ... increase steadily, and

$$d_1 = 5, \quad d_2 = 8, \quad d_3 = \tfrac{221}{25}, \quad d_4 = \tfrac{1517}{169}, \quad \ldots,$$

the limit of d_n being 9.

The nth Markoff form can be written

$$Q_n(u,v) = (u-\theta_n v)(u-\phi_n v), \tag{61}$$

where θ_n, ϕ_n are certain conjugate quadratic irrationals, depending on n, and

$$|\theta_n-\phi_n| = \sqrt{d_n}. \tag{62}$$

The minimum of $|Q_n(u,v)|$ is 1. It is known that $Q_n(u,v)$ assumes each of the values 1 and -1, and does so infinitely often, with values of u, v for which either factor in (61) becomes arbitrarily small.‡

We now investigate the circumstances under which the inequality (60) has, or has not, infinitely many solutions. For this we need to quote one further result (Koksma 1936), namely, that if θ is any irrational number which is not equivalent§ to any of θ_1, θ_2, ..., then there are, for any $\epsilon>0$, infinitely many pairs of integers u, v ($v>0$) for which

$$|(u-\theta v)v| < \tfrac{1}{3}+\epsilon. \tag{63}$$

† For references, see Koksma (1936, Kap. III, § 2). A comparatively simple proof of Markoff's theorem is given in Cassels (1949).

‡ For the fact that $Q_n(u, v)$ assumes both 1 and -1, see, for example, Dickson (1930, last clause of theorem 74). The rest of our assertion follows by the classical theory of the periodic chain of reduced forms equivalent to a given form.

§ Two irrational numbers θ, θ' are said to be equivalent if $\theta = (a\theta'+b)/(c\theta'+d)$, where a, b, c, d are integers with $ad-bc = \pm 1$; or, what is the same thing, if the continued fractions for θ and θ' have the same 'tails'. It is known that θ_n and ϕ_n in (61) are equivalent.

This is a classical result, and is proved by Markoff's methods, based on the theory of continued fractions.

Now consider any indefinite form $Q(u,v)$, which we can write, after removing a constant factor, as

$$Q(u,v) = (u-\theta v)(u-\phi v),$$

where

$$|\theta-\phi| = \surd d. \tag{64}$$

We shall suppose that θ and ϕ are both irrational; if one of them is rational, or both are rational, (60) has a trivial infinity of solutions.

Suppose first that θ is not equivalent to any of $\theta_1, \theta_2, \ldots$. Since

$$Q(u,v) = (\theta-\phi)(u-\theta v)v+(u-\theta v)^2,$$

it follows from (63) and (64) that the inequality

$$|Q(u,v)| < \tfrac{1}{3}\surd d+\epsilon$$

has infinitely many solutions for any $\epsilon > 0$. The number on the right can be made less than $\surd(d/d_n)$ for any given n, by suitable choice of ϵ. Hence, in this case, (60) has an infinity of solutions for any given n.

Suppose next that θ is equivalent to θ_n. After an integral unimodular substitution on u, v, and the removal of a constant factor, we can write

$$Q(u,v) = (u-\theta_n v)(u-\phi' v),$$

where

$$|\theta_n-\phi'| = \surd d.$$

We have

$$Q(u,v) = Q_n(u,v)\left(\frac{u-\phi' v}{u-\phi_n v}\right)$$

$$= Q_n(u,v)\left\{\frac{\theta_n-\phi'}{\theta_n-\phi_n}+\frac{(\phi'-\phi_n)\,t}{(\theta_n-\phi_n)(\theta_n-\phi_n+t)}\right\}, \tag{65}$$

where

$$t = \frac{u}{v}-\theta_n.$$

Now let u, v assume integral values, with $v \neq 0$, for which $Q_n(u,v) = \pm 1$, and for which $u-\theta_n v$ is arbitrarily small. In (65),

$$\left|\frac{\theta_n-\phi'}{\theta_n-\phi_n}\right| = \sqrt{\frac{d}{d_n}},$$

and t is arbitrarily small. Also

$$Q_n(u,v) = v^2\left(\frac{u}{v}-\theta_n\right)\left(\frac{u}{v}-\phi_n\right) = v^2 t(\theta_n-\phi_n+t).$$

Hence the additional term in the expression in brackets in (65) takes both positive and negative values as $Q_n(u,v)$ takes both the values 1 and -1, unless $\phi' = \phi_n$, in which case the additional term is zero. It follows that if θ is equivalent to θ_n, the inequality (60) has infinitely many solutions with strict inequality, unless the *form* $Q(u,v)$ is equivalent to a multiple of the *form* $Q_n(u,v)$, in which case there are no solutions with strict inequality.

Putting together the results proved above, we obtain the following:

THEOREM 10. *Let $Q(u,v)$ be any indefinite binary quadratic form with irrational roots θ, ϕ. Suppose that one at least of θ, ϕ is not equivalent to any of $\theta_1, \ldots, \theta_{n-1}$. Then the inequality*

$$|Q(u,v)| \leqslant \sqrt{\frac{d}{d_n}} \tag{66}$$

has infinitely many solutions, with the corresponding linear factor of $Q(u,v)$ *arbitrarily small; further, the inequality* (66) *has infinitely many such solutions with strict inequality unless the form* Q *is equivalent to a multiple of the form* Q_n, *in which case there are no solutions with strict inequality* (*except* $u = v = 0$).

8. Isolation theorems for XYZ

Let X, Y, Z be real linear forms in u, v, w of determinant 1. The available knowledge concerning the 'successive minima' of the product XYZ is much less than the corresponding knowledge, provided by Markoff's theorem, for the product of two linear forms in two variables. Only the first two minima are known, and we quote the result in the form of a lemma.†

Lemma 5. *Let* X, Y, Z *be real linear forms in* u, v, w *of determinant* 1. *Either* (1) *the product* XYZ *is equivalent to*

$$\tfrac{1}{7}(u+\theta v+\theta^2 w)\,(u+\phi v+\phi^2 w)\,(u+\psi v+\psi^2 w), \tag{67}$$

where θ, ϕ, ψ *are the roots of* $t^3+t^2-2t-1=0$, *in which case its minimum is* $\frac{1}{7}$; *or* (2) *the product is equivalent to*

$$\tfrac{1}{9}(u+\theta v+\theta^2 w)\,(u+\phi v+\phi^2 w)\,(u+\psi v+\psi^2 w), \tag{68}$$

where θ, ϕ, ψ *are the roots of* $t^3-3t-1=0$, *in which case its minimum is* $\frac{1}{9}$; *or* (3) *the inequality*

$$|XYZ| < \frac{1}{9\cdot 1} \tag{69}$$

is soluble.

It is important for what follows to observe‡ that in each of the cases referred to in the enunciation, the field $k(\theta)$ is galoisian and cyclic. The numbers 1, θ, θ^2 form a basis for the integers of $k(\theta)$. θ itself is a unit, with norm 1. The numerical values of θ, ϕ, ψ are

$$\theta = 1\cdot 246\ldots,\quad \phi = -0\cdot 445\ldots,\quad \psi = -1\cdot 801\ldots, \tag{70}$$

in the first case, and

$$\theta = 1\cdot 879\ldots,\quad \phi = -0\cdot 347\ldots,\quad \psi = -1\cdot 532\ldots, \tag{71}$$

in the second case. We denote conjugates by accents, the order being fixed by $\theta' = \phi$, $\theta'' = \psi$. Owing to the cyclic nature of the field, we have

$$(\alpha')' = \alpha'', \quad (\alpha'')' = \alpha$$

for any element α of $k(\theta)$.

For the present investigation we need the following three lemmas, which are valid for each of the two cubic fields defined above.

Lemma 6. *There exists an absolute constant* $A > 1$ *with the following property. For any positive numbers* λ *and* μ *there exists a unit* ω *in the field* $k(\theta)$ *such that*

$$\frac{\lambda}{A} < \pm\omega < A\lambda, \tag{72}$$

$$\frac{\mu}{A} < \pm\omega' < A\mu, \tag{73}$$

$$\frac{1}{A\lambda\mu} < \pm\omega'' < \frac{A}{\lambda\mu}, \tag{74}$$

where the three signs may be prescribed arbitrarily.

† The result is due to Davenport (see Davenport 1943).

‡ For these facts, see Davenport (1943).

Proof. It suffices to prove that there exists a constant B such that, for any positive λ, μ, there is a unit ω satisfying

$$\frac{1}{B}\lambda < \pm\omega < B\lambda, \tag{75}$$

$$\frac{1}{B}\mu < \pm\omega' < B\mu, \tag{76}$$

where the signs may be arbitrarily prescribed, and where also the sign of ω'' may be arbitrarily prescribed. For then

$$\frac{1}{B^2\lambda\mu} < |\,\omega''\,| < \frac{B^2}{\lambda\mu},$$

and the lemma then follows with $A = B^2$.

We take $\omega = \pm\phi^r\psi^s$, where r, s are integers. The signs of ω, ω', ω'' are

$$\pm(-)^r(-)^s, \quad \pm(-)^r, \quad \pm(-)^s,$$

respectively, and these may be given prescribed values by proper choice of the parities of r and s, and proper choice of $\pm$. It therefore suffices to satisfy (75) and (76) with $|\,\omega\,|$ and $|\,\omega'\,|$, subject to

$$r \equiv r_0 \pmod 2, \quad s \equiv s_0 \pmod 2. \tag{77}$$

Write $\qquad l_1 = \log|\,\theta\,|, \quad l_2 = \log|\,\phi\,|, \quad l_3 = \log|\,\psi\,|.$

The proposed inequalities for $|\,\omega\,|$ and $|\,\omega'\,|$ can be written

$$L - C < l_2 r + l_3 s < L + C, \quad M - C < l_3 r + l_1 s < M + C,$$

where $L = \log\lambda$, $M = \log\mu$, $C = \log B$. We have to prove the existence of C such that these inequalities are soluble for all λ, μ in integers r, s satisfying (77).

The linear forms $\qquad R = l_2 r + l_3 s, \quad S = l_3 r + l_1 s$

have a non-zero determinant. For if $l_2 = kl_3$ and $l_3 = kl_1$, then

$$0 = l_1 + l_2 + l_3 = l_1(1 + k + k^2),$$

which is impossible since k is real and $l_1 \neq 0$. The points (R, S) which correspond to integral values of r, s satisfying (77) constitute either a fixed lattice or one of three translations of this fixed lattice. It is therefore clear that there exists a constant $C > 0$ such that every square

$$L - C < R < L + C, \quad M - C < S < M + C$$

contains one of the points (R, S) in question. This proves the result.

LEMMA 7. *Let δ be a sufficiently small positive number. Let α, β, γ be a cyclic permutation of θ, ϕ, ψ. Let x, y, z be the linear forms*

$$\left.\begin{aligned} x &= u + (\alpha + \rho_1)\,v + (\alpha^2 + \rho_2)\,w, \\ y &= u + (\beta + \sigma_1)\,v + (\beta^2 + \sigma_2)\,w, \\ z &= u + (\gamma + \tau_1)\,v + (\gamma^2 + \tau_2)\,w, \end{aligned}\right\} \tag{78}$$

where ρ_1, ρ_2, σ_1, σ_2, τ_1, τ_2 are all numerically less than δ. Suppose ρ_1, ρ_2, σ_1, σ_2 are not all zero. Then there exist integers u, v, w, not all zero, such that

$$|\,xyz\,| < 1 - \delta_1, \quad |\,x\,| < 1, \quad |\,y\,| < 1, \tag{79}$$

where δ_1 is a positive absolute constant.

Proof. Since α, β, γ is a cyclic permutation of θ, ϕ, ψ there exist, for any unit ω of $k(\theta)$, rational integers u, v, w such that

$$\omega = u+\alpha v+\alpha^2 w, \quad \omega' = u+\beta v+\beta^2 w, \quad \omega'' = u+\gamma v+\gamma^2 w. \tag{80}$$

Solving these equations, we obtain linear expressions for v and w in terms of ω, ω', ω''. Substituting in (78), we get

$$\left.\begin{aligned} x &= \omega+\rho_3\omega+\rho_4\omega'+\rho_5\omega'', \\ y &= \omega'+\sigma_3\omega+\sigma_4\omega'+\sigma_5\omega'', \\ z &= \omega''+\tau_3\omega+\tau_4\omega'+\tau_5\omega''. \end{aligned}\right\} \tag{81}$$

The coefficients here are given by

$$\left.\begin{aligned} \Delta\rho_3 &= (\beta^2-\gamma^2)\,\rho_1-(\beta-\gamma)\,\rho_2, \\ \Delta\rho_4 &= (\gamma^2-\alpha^2)\,\rho_1-(\gamma-\alpha)\,\rho_2, \\ \Delta\rho_5 &= (\alpha^2-\beta^2)\,\rho_1-(\alpha-\beta)\,\rho_2, \end{aligned}\right\} \tag{82}$$

and by similar formulae for σ_3, σ_4, σ_5, τ_3, τ_4, τ_5, where

$$\Delta = \begin{vmatrix} 1, & \alpha, & \alpha^2 \\ 1, & \beta, & \beta^2 \\ 1, & \gamma, & \gamma^2 \end{vmatrix},$$

so that $|\Delta| = 7$ or 9. Since all the coefficients on the right of (82) are numerically less than 7, it follows that $\rho_3, \ldots, \tau_5$ in (81) are all numerically less than 2δ. We rewrite (81) as

$$\left.\begin{aligned} x/\omega &= 1+\rho_3+\rho_4\omega'/\omega+\rho_5\omega''/\omega, \\ y/\omega' &= 1+\sigma_4+\sigma_3\omega/\omega'+\sigma_5\omega''/\omega', \\ z/\omega'' &= 1+\tau_5+\tau_3\omega/\omega''+\tau_4\omega'/\omega''. \end{aligned}\right\} \tag{83}$$

Case 1. *Suppose that ρ_5 and σ_5 are not both zero.* By lemma 6, with $\mu = \lambda$, we can choose ω so that

$$\rho_5\omega''/\omega \leqslant 0, \quad \sigma_5\omega''/\omega' \leqslant 0,$$

$$\frac{\lambda}{A} < |\omega| < A\lambda, \quad \frac{\lambda}{A} < |\omega'| < A\lambda, \quad \frac{1}{A\lambda^2} < |\omega''| < \frac{A}{\lambda^2}, \tag{84}$$

where λ is any positive number. Then

$$1-2\delta-2\delta A^2-\frac{A^2}{\lambda^3}|\rho_5| < x/\omega < 1+2\delta+2\delta A^2-\frac{|\rho_5|}{A^2\lambda^3},$$

$$1-2\delta-2\delta A^2-\frac{A^2}{\lambda^3}|\sigma_5| < y/\omega' < 1+2\delta+2\delta A^2-\frac{|\sigma_5|}{A^2\lambda^3},$$

and
$$|z/\omega''| < 1+2\delta+2\delta A^2\lambda^3+2\delta A^2\lambda^3.$$

Suppose that $|\rho_5| \geqslant |\sigma_5|$; the proof in the other case is precisely similar. Choose λ so that

$$\frac{A^2}{\lambda^3}|\rho_5| = 1. \tag{85}$$

Then
$$\left.\begin{aligned}|x/\omega|<1+2\delta(1+A^2)-A^{-4},\\ |y/\omega'|<1+2\delta(1+A^2),\end{aligned}\right\} \tag{86}$$

and since
$$\lambda^3<2\delta A^2$$

by (85), we have
$$|z/\omega''|<1+2\delta+8\delta^2A^4.$$

If δ is sufficiently small, it follows from these inequalities that
$$|xyz|<1-\tfrac{1}{2}A^{-4}.$$

It follows from (86) and (84) that $|x|$ and $|y|$ are both less than
$$\{1+2\delta(1+A^2)\}A\lambda<\{1+2\delta(1+A^2)\}A\sqrt[3]{(2\delta A^2)}.$$

If δ is sufficiently small, this number is less than 1, and we have obtained (79).

Case 2. *Suppose that* $\rho_5=\sigma_5=0$. By (82) and the corresponding relations for σ_3, σ_4, σ_5 it is impossible that $\rho_4=\sigma_3=0$, since this would imply $\rho_1=\rho_2=\sigma_1=\sigma_2=0$, contrary to hypothesis. By (83),
$$x/\omega=1+\rho_3+\rho_4\omega'/\omega,\quad y/\omega'=1+\sigma_4+\sigma_3\omega/\omega',\quad z/\omega''=1+\tau_5+\tau_3\omega/\omega''+\tau_4\omega'/\omega''.$$

Suppose that $\rho_4\neq 0$; the proof when $\sigma_3\neq 0$ is precisely similar.

By lemma 6, with $\lambda=\mu^2$, we can choose ω so that
$$\rho_4\omega'/\omega<0,$$
$$\frac{\mu^2}{A}<|\omega|<A\mu^2,\quad \frac{\mu}{A}<|\omega'|<A\mu,\quad \frac{1}{A\mu^3}<|\omega''|<\frac{A}{\mu^3}, \tag{87}$$

where μ is any positive number. Then
$$1-2\delta-|\rho_4|\frac{A^2}{\mu}<x/\omega<1+2\delta-\frac{|\rho_4|}{A^2\mu},$$
$$|y/\omega'|<1+2\delta+2\delta A^2\mu,$$
$$|z/\omega''|<1+2\delta+2\delta A^2\mu^5+2\delta A^2\mu^4.$$

Choose μ so that
$$|\rho_4|\frac{A^2}{\mu}=1,$$

which implies
$$\mu<2\delta A^2.$$

Then
$$|x/\omega|<1+2\delta-A^{-4},$$
$$|y/\omega'|<1+2\delta+(2\delta A^2)^2,$$
$$|z/\omega''|<1+2\delta+(2\delta A^2)^6+(2\delta A^2)^5.$$

Thus, if δ is sufficiently small, we have
$$|xyz|<1-\tfrac{1}{2}A^{-4}.$$

Also, by (87), $|x|$ and $|y|$ are both less than
$$\{1+2\delta+(2\delta A^2)^2\}A\mu<\{1+2\delta+(2\delta A^2)^2\}2\delta A^3.$$

If δ is sufficiently small, this is less than 1, and we have obtained (79).

Lemma 8. *Let the linear forms x, y, z be as defined in lemma 7. Suppose now that not all of ρ_1, ρ_2, σ_1, σ_2, τ_1, τ_2 are zero. Then there exist integers u, v, w, not all zero, such that*

$$|xyz| < 1-\delta_1, \quad |x| < 1, \tag{88}$$

where δ_1 is a positive absolute constant.

Proof. By lemma 7, we must have $\rho_1 = \rho_2 = \sigma_1 = \sigma_2 = 0$, since otherwise we could satisfy

$$|xyz| < 1-\delta_1, \quad |x| < 1, \quad |y| < 1.$$

Also, by applying lemma 7 to the forms z, x, y, we must have $\tau_1 = \tau_2 = \rho_1 = \rho_2 = 0$, since otherwise we could satisfy

$$|xyz| < 1-\delta_1, \quad |z| < 1, \quad |x| < 1.$$

This proves lemma 8.

Theorem 11. *Let X, Y, Z be real linear forms in u, v, w of determinant 1. Either (1) the product XY is equivalent, by a unimodular substitution on u, v, w, to a multiple of*

$$(u+\theta v+\theta^2 w)\,(u+\phi v+\phi^2 w), \tag{89}$$

where θ, ϕ, ψ are either the roots of $t^3+t^2-2t-1=0$ or of $t^3-3t-1=0$, or (2) the inequalities

$$|XYZ| < \frac{1}{9\cdot 1}, \quad |X| < \epsilon, \quad |Y| < \epsilon \tag{90}$$

are soluble for any $\epsilon > 0$. In case (1), the inequalities (90) have only a finite number of solutions for each ϵ.

Proof. Let M denote the lower bound of the numbers λ such that the inequalities

$$|XYZ| < \lambda, \quad |X| < \mu, \quad |Y| < \mu,$$

are soluble for every $\mu > 0$ in integers u, v, w, not all zero. Such numbers λ exist, for, by theorem 5, any λ greater than $\frac{1}{7}$ has the property.

If $M < 1/9\cdot 1$, the final assertion of the theorem holds; hence we may suppose that

$$M \geq \frac{1}{9\cdot 1}. \tag{91}$$

By the definition of M, for every positive integer n there is a positive number μ_n such that the simultaneous inequalities

$$|XYZ| < M\left(1-\frac{1}{n}\right), \quad |X| < \mu_n, \quad |Y| < \mu_n, \tag{92}$$

are insoluble. Again, by the definition of M, the simultaneous inequalities

$$|XYZ| < M\left(1+\frac{1}{n}\right), \quad |X| < \frac{1}{n}\mu_n, \quad |Y| < \frac{1}{n}\mu_n, \tag{93}$$

are soluble for every n. Let X^*, Y^*, Z^* (depending on n) correspond to a solution of (93), so that

$$M\left(1-\frac{1}{n}\right) \leq |X^*Y^*Z^*| < M\left(1+\frac{1}{n}\right), \quad |X^*| < \frac{1}{n}\mu_n, \quad |Y^*| < \frac{1}{n}\mu_n. \tag{94}$$

Write $$x=\frac{X}{X^*},\quad y=\frac{Y}{Y^*},\quad z=\frac{Z}{Z^*};\tag{95}$$

these are linear forms in u, v, w, depending on n. Their determinant Δ_n, taken absolutely, is $|X^*Y^*Z^*|^{-1}$, and so satisfies
$$\frac{n}{n+1}\frac{1}{M}<\Delta_n\leqslant\frac{n}{n-1}\frac{1}{M},\tag{96}$$
by (94). From the insolubility of (92), it follows that if u, v, w are any integers, not all zero, then either
$$|xyz|>\frac{n-1}{n+1}\quad\text{or}\quad |x|>n\quad\text{or}\quad |y|>n.$$

The points (x, y, z) which correspond to integral values of u, v, w form a lattice Λ_n of determinant Δ_n which has a point at $(1, 1, 1)$ and has no point except O in the region defined by
$$|xyz|\leqslant\frac{n-1}{n+1},\quad |x|\leqslant n,\quad |y|\leqslant n.\tag{97}$$
Since the lattices Λ_n have bounded determinants, and have no point other than O in some fixed neighbourhood of O, it follows from Mahler's basic theorem that this sequence of lattices contains a subsequence which converges to a certain lattice Λ. The limit lattice Λ has determinant $1/M$, by (96), has a point at $(1, 1, 1)$, and has no point other than O satisfying $|xyz|<1$.

Let x, y, z denote, for the moment, the linear forms which correspond to the lattice Λ. The forms $M^{\frac{1}{3}}x$, $M^{\frac{1}{3}}y$, $M^{\frac{1}{3}}z$ have determinant 1, and their product is never less than M (except when $x=y=z=0$), where $M\geqslant 1/9{\cdot}1$, by (91). It follows from lemma 5 that $M=\frac{1}{7}$ or $\frac{1}{9}$, and that the product $Mxyz$ is equivalent to one of the two special products in (67) and (68). Hence, after a unimodular substitution on u, v, w, the linear forms x, y, z are given, in some order, by
$$A(u+\theta v+\theta^2w),\quad B(u+\phi v+\phi^2w),\quad C(u+\psi v+\psi^2w),\tag{98}$$
where A, B, C are constants and θ, ϕ, ψ have the two possibilities set out in lemma 5. By comparison of determinants, $ABC=\pm 1$.

Since $(1, 1, 1)$ is a point of Λ, there exist integers U, V, W for which the three expressions (98) all become 1. Since $ABC=\pm 1$, the number $\omega=U+\theta V+\theta^2W$ is a unit of $k(\theta)$. As the numbers $1, \theta, \theta^2$ form a basis of $k(\theta)$, the numbers of the form $\omega^{-1}(u+\theta v+\theta^2 w)$ are the integers of $k(\theta)$, and so there is an integral unimodular substitution from u, v, w to u', v', w' such that
$$u+\theta v+\theta^2w=\omega(u'+\theta v'+\theta^2w'),$$
$$u+\phi v+\phi^2w=\omega'(u'+\phi v'+\phi^2w'),$$
$$u+\psi v+\psi^2w=\omega''(u'+\psi v'+\psi^2w'),$$
where ω' and ω'' are the conjugates of ω in $k(\phi)$ and $k(\psi)$. Omitting the accents we can now write†
$$x=u+\alpha v+\alpha^2w,\quad y=u+\beta v+\beta^2w,\quad z=u+\gamma v+\gamma^2w,$$
where α, β, γ is a permutation of θ, ϕ, ψ. Since an interchange of X and Y in theorem 11 does not affect either the hypothesis or the conclusion, we can suppose that α, β, γ is a cyclic permutation of θ, ϕ, ψ.

† This result can also be deduced, using the fact that (1, 1, 1) is a lattice point, from the proof of lemma 5 given in Davenport (1943).

Having determined the nature of the lattice Λ, we now return to the lattice Λ_n, which converges to Λ as $n \to \infty$ through some sequence of positive integers. The lattice Λ_n is therefore given by

$$x = u + (\alpha+\rho_1)\,v + (\alpha^2+\rho_2)\,w,$$
$$y = u + (\beta+\sigma_1)\,v + (\beta^2+\sigma_2)\,w,$$
$$z = u + (\gamma+\tau_1)\,v + (\gamma^2+\tau_2)\,w,$$

where the inequalities

$$|\rho_1| < \delta, \quad |\rho_2| < \delta, \quad |\sigma_1| < \delta, \quad |\sigma_2| < \delta, \quad |\tau_1| < \delta, \quad |\tau_2| < \delta$$

are all satisfied, for any $\delta > 0$, if n is a sufficiently large number of the sequence.

Applying lemma 7 to the above forms, we deduce that either

$$\rho_1 = \rho_2 = \sigma_1 = \sigma_2 = 0,$$

or there exists a point other than O of Λ_n for which

$$|xyz| < 1-\delta_1, \quad |x| < 1, \quad |y| < 1,$$

where δ_1 is a positive absolute constant. This latter possibility cannot arise if n is sufficiently large, since no point of Λ_n other than O satisfies (97).

We have now proved that one of the lattices Λ_n is of the form

$$x = u+\alpha v+\alpha^2 w, \quad y = u+\beta v+\beta^2 w, \quad z = u+(\gamma+\tau_1)\,v+(\gamma^2+\tau_2)\,w,$$

where α, β, γ is a cyclic permutation of θ, ϕ, ψ, where these have one of the two possibilities of lemma 5. Returning to (95), it follows that XY is equivalent to a multiple of one of the three products which are obtained by multiplying together two of the three linear forms

$$u+\theta v+\theta^2 w, \quad u+\phi v+\phi^2 w, \quad u+\psi v+\psi^2 w. \tag{99}$$

The three possible products are themselves mutually equivalent. This follows from the fact that the field is galoisian and cyclic, so that there is a unimodular substitution which has the effect of permuting cyclically the three linear forms (99). Hence XY is equivalent to a multiple of (89), and this establishes the main assertion of theorem 11.

It remains to prove the final assertion of theorem 11. For this we may consider three forms X, Y, Z, where

$$X = u+\theta v+\theta^2 w, \quad Y = u+\phi v+\phi^2 w,$$

and where Z is any form such that the determinant of X, Y, Z is 1. We have to prove that the inequalities (90) have only a finite number of solutions for any $\epsilon > 0$. We shall suppose that θ, ϕ, ψ are the roots of $t^3+t^2-2t-1 = 0$, and we shall prove that the inequalities

$$|XYZ| < \kappa, \quad |X| < \epsilon, \quad |Y| < \epsilon, \tag{100}$$

have only a finite number of solutions for any $\epsilon > 0$, where κ is any fixed number less than $\frac{1}{7}$. The same proof applies in the other case, with $\frac{1}{9}$ in place of $\frac{1}{7}$.

The argument is a familiar one. Since any four linear forms in u, v, w are linearly dependent, there exists an identity

$$Z = c(u+\psi v+\psi^2 w)+aX+bY,$$

where a, b, c are constants. By comparison of determinants, $c = \pm\frac{1}{7}$. Hence

$$\begin{aligned} |XYZ| &\geqslant \tfrac{1}{7}\,|(u+\theta v+\theta^2 w)\,(u+\phi v+\phi^2 w)\,(u+\psi v+\psi^2 w)| - |XY(aX+bY)| \\ &\geqslant \tfrac{1}{7} - |XY(aX+bY)|, \end{aligned} \tag{101}$$

provided u, v, w are not all zero. If (100) had infinitely many solutions, there would be some with $|Z|$ arbitrarily large, and consequently $|XY|$ arbitrarily small. But then, since $|X|$ and $|Y|$ are bounded, we would have $|XY(aX+bY)|$ arbitrarily small, and (101) would contradict the first half of (100), if κ is a fixed number less than $\frac{1}{7}$. This establishes the result in question, and the proof of theorem 11 is complete.

Theorem 12. *Let X, Y, Z be real linear forms in u, v, w of determinant* 1. *Either* (1) *the product XYZ is equivalent to a multiple of*

$$(u+\theta v+\theta^2 w)\,(u+\phi v+\phi^2 w)\,(u+\psi v+\psi^2 w), \tag{102}$$

where θ, ϕ, ψ are either the roots of $t^3+t^2-2t-1=0$ or the roots of $t^3-3t-1=0$, or (2) *the inequalities*

$$|XYZ|<\frac{1}{9{\cdot}1}, \quad |X|<\epsilon \tag{103}$$

are soluble for any $\epsilon>0$.

Proof. This is substantially the same as the proof of theorem 11, except that we appeal to lemma 8 instead of to lemma 7. The interchange of two forms which is made in the course of the proof must now be effected with Y and Z.

Corollary. *Let X, Y, Z be real linear forms in u, v, w of determinant* 1. *The inequality*

$$|XYZ|<\frac{1}{9{\cdot}1}$$

has an infinity of solutions, unless the product XYZ is equivalent to a multiple of the product (102), *where θ, ϕ, ψ are either the roots of $t^3+t^2-2t-1=0$ or the roots of $t^3-3t-1=0$.*

A few comments on theorems 11 and 12 suggest themselves. It will be seen that in theorem 11 we prove, effectively, that except in the excluded cases, the inequality

$$|XYZ|<\frac{1}{9{\cdot}1}$$

has solutions in which *two* of $|X|$, $|Y|$, $|Z|$, prescribed at will, are arbitrarily small. The excluded cases are analogous to those in theorem 10, in that they relate to the corresponding linear factors and not to the product. On the other hand, by relaxing the subsidiary conditions to *one*, in theorem 12, the excluded cases are greatly simplified. This represents a possibility which obviously could not arise with a product of only two linear forms.

In theorem 12 we have not described the situation which arises when XYZ is equivalent to one of the special products (102). In this case, by obvious considerations with units, the *equation*

$$|XYZ| = \tfrac{1}{7} \text{ or } \tfrac{1}{9},$$

as the case may be, has infinitely many solutions, and among them are solutions in which two of $|X|$, $|Y|$, $|Z|$, prescribed at will, are arbitrarily small.

9. Isolation theorems for $(X^2+Y^2)\,Z$

We now investigate the possibility of obtaining for the product $(X^2+Y^2)\,Z$ similar results to those proved in § 8 for the product XYZ. Again X, Y, Z are real linear forms in u, v, w of determinant 1. Here the basis of knowledge from which we start is much slighter than it was there. Indeed, the known result of Davenport (1939), proved later more simply by Mordell (1942), that $\Delta(K) = \frac{1}{2}\surd(23)$ for the body K defined by $|(x^2+y^2)z| \leqslant 1$, only shows that the inequality

$$|(X^2+Y^2)Z| < \frac{2}{\surd(23)} + \epsilon$$

is soluble for every $\epsilon > 0$. We have, however, seen in theorem 3 (33*a*) that the inequality

$$|(X^2+Y^2)Z| < \frac{2}{\surd(23)} \tag{104}$$

has an infinity of solutions, unless $(X^2+Y^2)Z$ is equivalent to a multiple of the special product (40), which is

$$(u+\theta v+\theta^2 w)(u+\bar\theta v+\bar\theta^2 w)(u+\phi v+\phi^2 w),$$

where θ, $\bar\theta$ are the complex roots and ϕ the real root of $t^3-t-1 = 0$. Numerically,

$$\theta = -0{\cdot}662\ldots+i\,0{\cdot}562\ldots, \quad \phi = 1{\cdot}324\ldots.$$

The question of the isolation of the inequality (104) is one which must be formulated with some care (see Davenport 1941). The excluded case for the inequality (104) occurs when $(X^2+Y^2)Z$ is equivalent to

$$\frac{2}{\surd(23)}(P^2+Q^2)R,$$

where

$$P+iQ = u+\theta v+\theta^2 w, \quad R = u+\phi v+\phi^2 w. \tag{105}$$

The minimum of $|(P^2+Q^2)R|$ is 1. Now consider the possibility that $(X^2+Y^2)Z$ is equivalent to

$$\frac{2}{\surd\{23(1+\lambda^2+\mu^2)\}}(P^2+Q^2+(\lambda P+\mu Q)^2)R, \tag{106}$$

where λ and μ are real. As

$$P^2+Q^2+(\lambda P+\mu Q)^2 = (1+\lambda^2+\mu^2)P^2+(1+\lambda^2+\mu^2)Q^2-(\mu P-\lambda Q)^2,$$

it is clear that the minimum of the absolute value of (106) lies between the limits

$$\frac{2}{\surd\{23(1+\lambda^2+\mu^2)\}}, \quad 2\surd\left\{\frac{1+\lambda^2+\mu^2}{23}\right\}.$$

As these limits are arbitrarily near to $2/\surd(23)$ if λ and μ are arbitrarily small, the inequality (104) certainly cannot be isolated unless we exclude the possibility that $(X^2+Y^2)Z$ is equivalent to (106).

We shall, however, prove that the inequality (104) is in fact isolated, if the above possibility is excluded, and that this isolation still applies if an infinity of solutions are required. For this we need three lemmas. We denote by K the body in three-dimensional space defined by

$$|(x^2+y^2)z| \leqslant 1.$$

Lemma 9. *For any $\delta > 0$ there exists a number $\delta' > 0$ with the following property. Any lattice Λ which is admissible for K and has a point at* $(1, 0, 1)$, *and for which*

$$d(\Lambda) < \tfrac{1}{2}\surd(23)+\delta', \tag{107}$$

is given by

$$\left.\begin{aligned} x \pm iy &= u+(\theta+\rho_1)v+(\theta^2+\rho_2)w, \\ z &= u+(\phi+\sigma_1)v+(\phi^2+\sigma_2)w, \end{aligned}\right\} \tag{108}$$

where ρ_1, ρ_2 *are complex and* σ_1, σ_2 *are real, and*

$$|\rho_1|<\delta, \quad |\rho_2|<\delta, \quad |\sigma_1|<\delta, \quad |\sigma_2|<\delta. \tag{109}$$

Proof. Let Λ_0 be the lattice defined by

$$x+iy = u+\theta v+\theta^2 w, \quad z = u+\phi v+\phi^2 w,$$

and $\overline{\Lambda}_0$ that derived from it by changing y into $-y$. If the lemma is false, then for some $\delta>0$ there exists, for any $\delta'>0$, a lattice satisfying the hypotheses, which is not of the form stated in (108) and (109). Hence there is a sequence Λ_n of lattices, each admissible for K and having a point at $(1, 0, 1)$, such that $d(\Lambda_n)\to\frac{1}{2}\sqrt{(23)}$ and Λ_n is not of the form stated in (108) and (109). Such a sequence Λ_n has a convergent subsequence, converging to some limit lattice Λ'. Then Λ' is a critical lattice of K and has a point at $(1, 0, 1)$. We prove that $\Lambda' = \Lambda_0$ or $\overline{\Lambda}_0$.

As Λ' is a critical lattice of K with a point on the boundary of K, it follows by the result of Mordell stated on p. 318 that $\Lambda' = \Omega\Lambda_0$ for some automorphism of K. This means that Λ' is given by

$$x\pm iy = \lambda(u+\theta v+\theta^2 w), \quad z = \mu(u+\phi v+\phi^2 w),$$

where λ is complex, μ is real and $|\lambda^2\mu| = 1$. We know that there exist integers U, V, W such that

$$1 = \lambda(U+\theta V+\theta^2 W), \quad 1 = \mu(U+\phi V+\phi^2 W).$$

Since $|\lambda^2\mu| = 1$, it follows that the number $\omega = U+\theta V+\theta^2 W$ is a unit in the field $k(\theta)$. As the numbers 1, θ, θ^2 form a basis of $k(\theta)$, the numbers of the form $\omega^{-1}(u+\theta v+\theta^2 w)$ are the integers of $k(\theta)$ and there is an integral unimodular substitution from u, v, w to u', v', w' such that

$$u+\theta v+\theta^2 w = \omega(u'+\theta v'+\theta^2 w'),$$

$$u+\overline{\theta} v+\overline{\theta}^2 w = \overline{\omega}(u'+\overline{\theta} v'+\overline{\theta}^2 w'),$$

$$u+\phi v+\phi^2 w = \omega'(u'+\phi v'+\phi^2 w'),$$

where $\overline{\omega}$ and ω' are the conjugates of ω in $k(\overline{\theta})$ and $k(\phi)$. Omitting the accents we can now write

$$x\pm iy = u+\theta v+\theta^2 w, \quad z = u+\phi v+\phi^2 w.$$

Thus either $\Lambda' = \Lambda_0$ or $\Lambda' = \overline{\Lambda}_0$.

Since a subsequence of the lattices Λ_n converges to Λ', it is clear that some member of this sequence is expressible in the form (108) where (109) is satisfied, contrary to the choice of this sequence. This contradiction proves the lemma.

Lemma 10. *Let* δ *be a sufficiently small positive number. Let* x, y, z *be the linear forms given in* (108), *where* ρ_1, ρ_2, σ_1, σ_2 *satisfy* (109). *Then* (1) *if* σ_1 *and* σ_2 *are not both zero, there exist integers* u, v, w, *not all zero, such that*

$$|(x^2+y^2)z|<1-\delta_1, \quad |z|<1, \tag{110}$$

where δ_1 *is a positive absolute constant. Also* (2) *if* ρ_1 *and* ρ_2 *do not satisfy*

$$\phi\rho_1+\rho_2 = 0, \tag{111}$$

there exist integers u, v, w, *not all zero, such that*

$$|(x^2+y^2)z|<1-\delta_1, \quad x^2+y^2<1. \tag{112}$$

Proof. We follow at first the same general lines as in the proof of lemma 7. For any unit ω of $k(\theta)$, there exist integers u, v, w such that

$$\omega = u+\theta v+\theta^2 w, \quad \bar{\omega} = u+\bar{\theta} v+\bar{\theta}^2 w, \quad \omega' = u+\phi v+\phi^2 w,$$

where ω' is the conjugate of ω in the real field $k(\phi)$. If we give u, v, w these values in (108), we obtain

$$x\pm iy = \omega+\rho_3\omega+\rho_4\bar{\omega}+\rho_5\omega', \quad z = \omega'+\sigma_3\omega+\sigma_4\bar{\omega}+\sigma_5\omega'. \tag{113}$$

The coefficients here are given, as in (82), by

$$\left.\begin{aligned}\Delta\rho_3 &= (\bar{\theta}^2-\phi^2)\,\rho_1-(\bar{\theta}-\phi)\,\rho_2,\\ \Delta\rho_4 &= (\phi^2-\theta^2)\,\rho_1-(\phi-\theta)\,\rho_2,\\ \Delta\rho_5 &= (\theta^2-\bar{\theta}^2)\,\rho_1-(\theta-\bar{\theta})\,\rho_2,\end{aligned}\right\} \tag{114}$$

and by similar formulae for $\sigma_3, \sigma_4, \sigma_5$, where

$$\Delta = \begin{vmatrix} 1, & \theta, & \theta^2 \\ 1, & \bar{\theta}, & \bar{\theta}^2 \\ 1, & \phi, & \phi^2 \end{vmatrix} = -i\surd(23).$$

The precise formulae are not important, so long as we observe that if $\sigma_3 = \sigma_4 = 0$ then $\sigma_1 = \sigma_2 = 0$, and if $\rho_5 = 0$ then (111) holds. The latter statement follows from the last formula of (114), since $\theta+\bar{\theta} = -\phi$. We note also that $\rho_3, \ldots, \sigma_5$ are all less than $C\delta$ in absolute value, where C is an absolute constant. Plainly σ_3 and σ_4 are complex conjugates, since z in (113) is real.

Case 1. *Suppose that σ_1 and σ_2 are not both zero.* We take $\omega = \theta^{-r}$, where r is an integer, positive or zero. Then $\omega' = \phi^{-r}$, and we have

$$\left|\frac{\bar{\omega}}{\omega}\right| = 1, \quad \left|\frac{\omega'}{\omega}\right| \leqslant 1.$$

It follows from (113) that

$$|(x\pm iy)/\omega| < 1+3C\delta, \tag{115}$$

$$z/\omega' = 1+\sigma_5+\sigma_3\left(\frac{\phi}{\theta}\right)^r+\sigma_4\left(\frac{\phi}{\bar{\theta}}\right)^r. \tag{116}$$

Write

$$\frac{\phi}{\theta} = k\,e^{i\chi}, \tag{117}$$

where $k>1$ and χ is real. Since σ_3 and σ_4 are complex conjugates, we have

$$\sigma_3\left(\frac{\phi}{\theta}\right)^r+\sigma_4\left(\frac{\phi}{\bar{\theta}}\right)^r = 2\sigma k^r\cos(r\chi+\alpha),$$

where $\sigma = |\sigma_3| = |\sigma_4| > 0$ (by the hypothesis of this case), and α is real. Hence, by (116),

$$1-C\delta+2\sigma k^r\cos(r\chi+\alpha) < z/\omega' < 1+C\delta+2\sigma k^r\cos(r\chi+\alpha). \tag{118}$$

In the notation of Davenport (1939), we have

$$\frac{\phi}{\theta} = \phi^2\bar{\theta} = \phi^2(-\tfrac{1}{2}\phi-i\psi).$$

Since $|\phi/\theta| = \phi^{\frac{3}{2}}$, the angle χ in (117) is given by

$$\cos\chi = -\tfrac{1}{2}\phi^{\frac{3}{2}} = -0{\cdot}76235\ldots, \quad \sin\chi < 0.$$

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Hence $$\chi = 220^\circ\, 19{\cdot}7'\ldots.$$

The angles $r\chi$, where $r = 0, 1, 2, 3, 4$, are

$$0^\circ, \quad 220^\circ\, 20', \quad 80^\circ\, 39', \quad 300^\circ\, 59', \quad 161^\circ\, 19',$$

approximately. The greatest interval between two of these angles is less than 82°. It follows that for any angle α there exists, in any set of five consecutive integers, one for which

$$|\, r\chi + \alpha - 180^\circ \,| < 41^\circ \quad (\mathrm{mod}\ 360^\circ),$$

that is, $$\cos\,(r\chi+\alpha) < -\cos 41^\circ < -\tfrac{3}{4}. \tag{119}$$

Choose r to be the largest integer satisfying (119) for which

$$2\sigma k^r \leqslant 1. \tag{120}$$

Then, owing to the result just stated, we have

$$2\sigma k^{r+5} > 1. \tag{121}$$

Since $\sigma < C\delta$, this implies that $$k^r > \frac{1}{2\sigma k^5} > \frac{1}{2Ck^5\delta}. \tag{122}$$

In particular, if δ is sufficiently small, the condition $r \geqslant 0$ is satisfied. Since, by (119),

$$-1 \leqslant \cos\,(r\chi+\alpha) < -\tfrac{3}{4},$$

it follows from (118) that $$z/\omega' > 1 - C\delta - 2\sigma k^r \geqslant -C\delta,$$

$$z/\omega' < 1 + C\delta - \tfrac{3}{4}(2\sigma k^r) < 1 + C\delta - \tfrac{3}{4}k^{-5}.$$

Hence $$|\, z/\omega' \,| < 1 + C\delta - \tfrac{3}{4}k^{-5}. \tag{123}$$

As k is an absolute constant, it follows from (115) and (123) that if δ is sufficiently small we have $$|\,(x^2+y^2)\,z\,| < 1 - \delta_1,$$

where δ_1 is a positive absolute constant.

Also, from (123) and the fact that $\omega' = \phi^{-r}$, where $\phi > 1$ and r satisfies (122), it is clear that $|\, z \,| < 1$ if δ is sufficiently small.

Case 2. *Suppose that* ρ_1 *and* ρ_2 *do not satisfy* (111). As we have seen earlier, this implies that $\rho_5 \neq 0$. We now take $\omega = \theta^r$, where r is an integer, positive or zero. It follows from (113) that

$$\frac{x \pm iy}{\omega} = 1 + \rho_3 + \rho_4 \left(\frac{\bar\theta}{\theta}\right)^r + \rho_5 \left(\frac{\phi}{\theta}\right)^r, \tag{124}$$

$$|\, z/\omega' \,| < 1 + 3C\delta. \tag{125}$$

We write $$\rho_5 = \rho\, e^{i\beta},$$

where $\rho > 0$ by hypothesis, and β is real. By (124) and (117),

$$\frac{x \pm iy}{\omega} = 1 + \rho_3 + \rho_4 \left(\frac{\bar\theta}{\theta}\right)^r + \rho k^r\, e^{i(r\chi+\beta)}.$$

Hence $$\left|\frac{x \pm iy}{\omega} - \rho_3 - \rho_4 \left(\frac{\bar\theta}{\theta}\right)^r\right|^2 = 1 + (\rho k^r)^2 + 2\rho k^r \cos\,(r\chi+\beta). \tag{126}$$

We choose r to be the largest integer satisfying

$$\cos(r\chi+\beta) < -\tfrac{3}{4} \tag{127}$$

for which

$$2\rho k^r \leqslant 1. \tag{128}$$

Then, as in the previous case, we have

$$2\rho k^{r+5} > 1, \tag{129}$$

and since $\rho < C\delta$, this implies that

$$k^r > \frac{1}{2\rho k^5} > \frac{1}{2Ck^5\delta}, \tag{130}$$

so that $r \geqslant 0$ if δ is sufficiently small. It follows from (126) and (127) that

$$\left|\frac{x\pm iy}{\omega} - \rho_3 - \rho_4\left(\frac{\bar\theta}{\theta}\right)^r\right|^2 < 1+(\rho k^r)^2 - \tfrac{3}{2}\rho k^r$$
$$\leqslant 1-\rho k^r,$$

since $\rho k^r \leqslant \frac{1}{2}$ by (128). The right-hand side is less than $1-\frac{1}{2}k^{-5}$, by (129). Hence

$$|(x+iy)/\omega| < 2C\delta + \sqrt{(1-\tfrac{1}{2}k^{-5})}. \tag{131}$$

It follows from (125) and (131) that if δ is sufficiently small, we have

$$|(x^2+y^2)\,z| < 1-\delta_1,$$

where δ_1 is a positive absolute constant. Also, since $\omega = \theta^r$, where $|\theta| < 1$ and r satisfies (130), we see that

$$x^2+y^2 < 1$$

if δ is sufficiently small. This completes the proof of lemma 10.

Lemma 11. *Let δ be a sufficiently small positive number. Let x, y, z be the linear forms given in* (108), *where ρ_1, ρ_2, σ_1, σ_2 satisfy* (109). *Suppose now that not all of the numbers $\phi\rho_1+\rho_2$, σ_1, σ_2 are zero. Then there exist integers u, v, w, not all zero, such that*

$$|(x^2+y^2)\,z| < 1-\delta_1, \tag{132}$$

where δ_1 is a positive absolute constant.

Proof. This lemma is an immediate corollary of lemma 10.

Theorem 13. *There exists a positive absolute constant δ_2 with the following property. Let X, Y, Z be real linear forms in u, v, w of determinant* 1. *Then, if Z is not equivalent to a multiple of*

$$R = u+\phi v+\phi^2 w, \tag{133}$$

the inequalities

$$|(X^2+Y^2)\,Z| < \frac{2}{\sqrt{(23)}}(1-\delta_2), \quad |Z| < \epsilon, \tag{134}$$

are soluble for any $\epsilon > 0$. Also, if X^2+Y^2 is not equivalent to a multiple of

$$P^2+Q^2+(\lambda P+\mu Q)^2, \tag{135}$$

where λ, μ are arbitrarily real numbers and P, Q are the real linear forms defined by

$$P+iQ = u+\theta v+\theta^2 w, \tag{136}$$

then the inequalities

$$|(X^2+Y^2)\,Z| < \frac{2}{\sqrt{(23)}}(1-\delta_2), \quad X^2+Y^2 < \epsilon, \tag{137}$$

are soluble for any $\epsilon > 0$.

Proof. This is on the same general lines as the proof of theorem 11. We content ourselves with proving the second half of the theorem, as the proof of the first half is similar but slightly simpler.

Choose any positive number δ so small that lemma 10 is satisfied. Let δ' be the corresponding number given by lemma 9. Choose any positive δ_2 so small that

$$\frac{\surd(23)}{2(1-\delta_2)}<\tfrac{1}{2}\surd(23)+\delta'. \tag{138}$$

We prove the required result with this value of δ_2.

Let M denote the lower bound of the numbers λ such that the inequalities

$$|(X^2+Y^2)Z|<\lambda, \quad X^2+Y^2<\mu$$

are soluble for every $\mu>0$. Such numbers λ exist, since by theorem 6 every λ greater than $2/\surd(23)$ has the property. If $M<2(1-\delta_2)/\surd(23)$, there is nothing to prove, so we may suppose that

$$M\geqslant\frac{2}{\surd(23)}(1-\delta_2). \tag{139}$$

As in the proof of theorem 11, for every positive integer n, we choose a number $\mu_n>0$ such that the inequalities

$$|(X^2+Y^2)Z|<M\left(1-\frac{1}{n}\right), \quad X^2+Y^2<\mu_n$$

are insoluble, and values X^*, Y^*, Z^* (depending on n) of X, Y, Z, such that

$$M\left(1-\frac{1}{n}\right)\leqslant|(X^{*2}+Y^{*2})Z^*|<M\left(1+\frac{1}{n}\right), \quad X^{*2}+Y^{*2}<\frac{\mu_n}{n}.$$

Write
$$x=\frac{XX^*+YY^*}{X^{*2}+Y^{*2}}, \quad y=\frac{XY^*-YX^*}{X^{*2}+Y^{*2}}, \quad z=\frac{Z}{Z^*}. \tag{140}$$

These are linear forms in u, v, w depending on n; their determinant Δ_n, taken absolutely, satisfies

$$\frac{n}{n+1}\frac{1}{M}<\Delta_n\leqslant\frac{n}{n-1}\frac{1}{M}.$$

The lattice Λ_n which corresponds to them has a point at $(1,0,1)$ and has no point other than O in the region

$$|(x^2+y^2)z|\leqslant\frac{n-1}{n+1}, \quad x^2+y^2\leqslant n.$$

Since the lattices Λ_n have bounded determinants, and have no point other than O in some fixed neighbourhood of O, it follows from Mahler's basic theorem that this sequence of lattices contains a subsequence which converges to a certain lattice Λ. The limit lattice Λ has determinant $1/M$, has a point at $(1,0,1)$ and is admissible for K. By (138) and (139)

$$d(\Lambda)=\frac{1}{M}<\tfrac{1}{2}\surd(23)+\delta'.$$

Thus Λ satisfies the conditions of lemma 9, and so Λ can be expressed in the form (108) where the inequalities (109) are satisfied. Hence, if we choose a lattice Λ_n of the subsequence converging to Λ with n sufficiently large, Λ_n will be expressible in the form (108) where the inequalities (109) are satisfied, and, further, there will be no point other than O of Λ_n in the region

$$|(x^2+y^2)z|<1-\delta_1, \quad x^2+y^2<1.$$

Thus, after applying an integral unimodular substitution to u, v, w, the forms x, y, z corresponding to this lattice Λ_n satisfy the conditions of lemma 10. Hence, by the second half of lemma 10,

$$x\pm iy = u+(\theta+\rho_1)v+(\theta^2+\rho_2)w,$$

where $\phi\rho_1+\rho_2=0$, and so

$$x\pm iy = u+(\theta+\rho_1)v+(\theta^2-\phi\rho_1)w$$
$$= P+iQ+\rho_1(v-\phi w).$$

Now

$$2iQ = (u+\theta v+\theta^2 w)-(u+\bar{\theta}v+\bar{\theta}^2 w)$$
$$= (\theta-\bar{\theta})(v-\phi w).$$

Consequently

$$x = P+\beta Q, \quad y = \alpha Q,$$

for some real α, β. It follows from (140) that X^2+Y^2 is equivalent to some positive definite quadratic form in P and Q, and so X^2+Y^2 is equivalent to a multiple of the form (135) for suitable values of λ and μ. This proves the second half of theorem 13.

THEOREM 14. *There exists a positive absolute constant δ_2 with the following property. Let X, Y, Z be real linear forms in u, v, w of determinant* 1. *Then either* (1) *the inequality*

$$|(X^2+Y^2)Z|<\frac{2}{\sqrt{(23)}}(1-\delta_2) \tag{141}$$

is soluble in integers u, v, w not all zero; *or* (2) *the product $(X^2+Y^2)Z$ is equivalent to the product*

$$\frac{2}{\sqrt{\{23(1+\lambda^2+\mu^2)\}}}\{P^2+Q^2+(\lambda P+\mu Q)^2\}R \tag{142}$$

or some real λ, μ, where P, Q, R are the real linear forms defined by

$$P+iQ = u+\theta v+\theta^2 w, \quad R = u+\phi v+\phi^2 w, \tag{143}$$

and in this case, for every $\epsilon>0$, the inequality

$$|(X^2+Y^2)Z|<\frac{2(1+\epsilon)}{\sqrt{\{23(1+\lambda^2+\mu^2)\}}} \tag{144}$$

is soluble in integers u, v, w not all zero.

Proof. The main clauses of the theorem may be proved by the method used to prove theorem 13. We need only remark that δ_2 is defined as in theorem 13, that M is defined to be the lower bound of the numbers ν such that the inequality

$$|(X^2+Y^2)Z|<\nu$$

is soluble, and that lemma 11 is used in place of lemma 10.

We consider now the case when $(X^2+Y^2)\,Z$ is equivalent to the form (142), and we suppose that λ and μ are not both zero. We use the notation of lemma 10. We first prove that there is no positive integer r such that θ^r is real. Suppose that $r \geqslant 1$ and θ^r is real. Then

$$\theta^r = \bar{\theta}^r,$$

and if $r \geqslant 3$ we can use the identities

$$\theta^3 = \theta+1, \quad \bar{\theta}^3 = \bar{\theta}+1$$

to deduce that

$$\theta^{r-2}+\theta^{r-3} = \bar{\theta}^{r-2}+\bar{\theta}^{r-3}.$$

Proceeding inductively in this way we find that

$$a\theta^2+b\theta+c = a\bar{\theta}^2+b\bar{\theta}+c$$

for some non-negative integers a, b, c with a and b not both zero. But as $\theta \neq \bar{\theta}$, this equation implies that

$$a\phi = b,$$

which is impossible as ϕ is not rational. Thus

$$\theta = |\,\theta\,|\,e^{-i\chi},$$

where χ is a real *irrational* multiple of π.

Now for any integer r we can choose integers u, v, w such that

$$P+iQ = u+\theta v+\theta^2 w = \theta^r, \quad R = u+\phi v+\phi^2 w = \phi^r.$$

Then

$$P^2+Q^2 = \theta^r\bar{\theta}^r, \quad (\lambda P+\mu Q)^2 = (\tfrac{1}{2}\lambda(\theta^r+\bar{\theta}^r)-\tfrac{1}{2}i\mu(\theta^r-\bar{\theta}^r))^2,$$

so that

$$\frac{(\lambda P+\mu Q)^2}{P^2+Q^2} = \frac{\lambda^2+\mu^2}{4}\left|\frac{\lambda+i\mu}{\lambda-i\mu}+e^{-2ir\chi}\right|^2.$$

Thus, for any $\epsilon > 0$, we can choose r so that

$$\frac{(\lambda P+\mu Q)^2}{P^2+Q^2} < \epsilon.$$

Then

$$|\,(P^2+Q^2+(\lambda P+\mu Q)^2)\,R\,| < |\,\theta^r\bar{\theta}^r\phi^r\,|\,(1+\epsilon) = 1+\epsilon,$$

and so there are integers u, v, w not all zero satisfying (144). This proves the last clause of the theorem.

It will be seen from theorem 13 that the inequality

$$|\,(X^2+Y^2)\,Z\,| < \frac{2}{\sqrt{(23)}}\,(1-\delta_2)$$

has infinitely many solutions unless Z is equivalent to a multiple of R *and* X^2+Y^2 is equivalent (but not necessarily by the same substitution) to a multiple of $P^2+Q^2+(\lambda P+\mu Q)^2$. It is not difficult to show, by using the method in the proof of the last part of theorem 11, that these exceptional cases are all essential (provided $1+\lambda^2+\mu^2 < (1-\delta_2)^{-2}$).

It should, perhaps, be pointed out that the method by which we have proved theorems 13 and 14 offers no hope of giving an explicit value for the 'absolute' constant δ_2. The difficulty lies with lemma 9. It may be conjectured that theorem 13 would remain valid as long as

$$\frac{2}{\sqrt{(23)}}\,(1-\delta_2) > \frac{2}{\sqrt{(31)}},$$

since -31 is the next discriminant of a complex cubic field after -23. At present, no method is known which offers hope of proving this; but possibly the fact that theorem 13 indicates the *existence* of a second minimum for this problem may lead to further work on the question.

The conjecture just referred to has a bearing on the important unsolved problem of simultaneous Diophantine approximation to two irrational numbers Θ, Φ. If c denotes the lower bound of the numbers c' such that the inequalities

$$\left|\frac{p}{r}-\Theta\right|^2<\frac{c'}{r^3}, \quad \left|\frac{q}{r}-\Phi\right|^2<\frac{c'}{r^3} \quad (r>0)$$

have infinitely many integral solutions for any Θ, Φ, it is known that

$$\frac{1}{\sqrt{(23)}}\leqslant c<\frac{2}{\sqrt{(23)}}.$$

The above conjecture, if true, would allow one to replace the upper estimate $\frac{2}{\sqrt{(23)}}$ by $\frac{2}{\sqrt{(31)}}$.

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